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Free semidefinite representation of matrix power functions

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ABSTRACT

Consider the matrix power function X^p defined over the cone of positive definite matrices $\mathcal{S}^n_{++}$. It is known that X^p is convex over $\mathcal{S}^n_{++}$ if $p \in [-1, 0] \cup [1, 2]$ and X^p is concave over $\mathcal{S}^n_{++}$ if $p \in [0, 1]$. We show that the hypograph of X^p admits a free semidefinite representation if $p \in [0, 1]$ is rational, and the epigraph of X^p admits a free semidefinite representation if $p \in [-1, 0] \cup [1, 2]$ is rational.

1. Introduction

Let $\mathcal{S}^n$ be the space of real symmetric $n \times n$ matrices, and $\mathcal{S}^n_+$ (resp. $\mathcal{S}^n_{++}$) be the cone of positive semidefinite (resp. definite) matrices in $\mathcal{S}^n$. For $p \in \mathbb{R}$, the matrix power function X^p on $\mathcal{S}^n$ is defined as $X^p = Q^T \Lambda^p Q$ when this makes sense, with $X = Q^T \Lambda Q$ an orthogonal spectral decomposition. It is well known (cf. [3, p. 147]) that

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(1) X^p is convex over $\mathcal{S}^n_{++}$ if $p \in [-1, 0] \cup [1, 2]$, and
(2) X^p is concave over $\mathcal{S}^n_{+}$ if $p \in [0, 1]$.

Here, the concavity and convexity are defined as usual for functions of matrices. The goal of this paper is to give a *free semidefinite representation* (i.e., in terms of linear matrix inequalities whose construction is independent of the matrix dimension n) for the epigraph or hypograph of the matrix power function X^p for a range of rational exponents p.

1.1. Convex and concave matrix-valued functions

Let $\mathcal{D}$ be a convex subset of the space of the Cartesian product $(\mathcal{S}^n)^g$, with $g > 0$ an integer. A matrix-valued function $f : \mathcal{D} \to \mathcal{S}^n$ is **convex** if

$$f\big(tX + (1-t)Y\big) \preceq tf(X) + (1-t)f(Y), \quad \forall t \in [0, 1]$$

for all $X, Y \in \mathcal{D}$. If $-f$ is convex, we say that f is **concave**. The **epigraph** (resp. **hypograph**) of f is then defined as

$$\big\{(X, Y) \in \mathcal{D} \times \mathcal{S}^n : \ f(X) \preceq Y\big\} \qquad \big(\textit{resp.} \quad \big\{(X, Y) \in \mathcal{D} \times \mathcal{S}^n : \ f(X) \succeq Y\big\}\big).$$

The following is a straightforward but useful fact. Due to lacking of a suitable reference in case of matrix-valued functions, we include a short proof here.

Lemma 1.1. *Suppose $\mathcal{D}$ is a convex set. Then f is convex over $\mathcal{D}$ if and only if its epigraph is convex. Similarly, f is concave over $\mathcal{D}$ if and only if its hypograph is convex.*

Proof. We will prove only the first half of the proposition as the second half clearly follows from the first.

($\Rightarrow$) If (X, W) and (Y, Z) are in the epigraph of f and if $t \in [0, 1]$, then by the convexity of f,

$$f\big(tX + (1-t)Y\big) \preceq tf(X) + (1-t)f(Y) \preceq tW + (1-t)Z$$

so that $(tX + (1-t)Y, tW + (1-t)Z)$ is in the epigraph of f.

($\Leftarrow$) If $X, Y \in \mathcal{D}$ and $t \in \mathbb{R}$, then $(X, f(X))$ and $(Y, f(Y))$ are in the epigraph of f. Since the epigraph is convex, $(tX + (1-t)Y, \ tf(X) + (1-t)f(Y))$ is in the epigraph as well. But this says that $f(tX + (1-t)Y) \preceq tf(X) + (1-t)f(Y)$. $\square$

In the case that $f(X) \succeq 0$ for all $X \in \mathcal{D}$, we are often only interested in the pairs (X, Y) from the hypograph of f with $Y \succeq 0$. Thus, in this case, we slightly abuse terminology and refer to

$$\big\{(X, Y) \in \mathcal{D} \times \mathcal{S}^n_+ : \ f(X) \succeq Y\big\}$$