

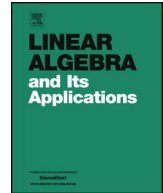


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Superadditivity and derivative of operator functions



Mitsuru Uchiyama^{a,1}, Atsushi Uchiyama^{b,*}, Mariko Giga^c

^a Department of Mathematics, Interdisciplinary Faculty of Science and Engineering, Shimane University, Matsue city, Shimane 690-8504, Japan

^b Department of Mathematical Sciences, Faculty of Science, Yamagata University, Yamagata city, Yamagata 990-8560, Japan

^c Department of Mathematics, Nippon Medical School, Kawasaki city, Kanagawa 221-0063, Japan

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ABSTRACT

We will show that if $\sum_{i \neq j} A_i A_j \geq 0$ for bounded operators $A_i \geq 0$ ($i = 1, 2, \dots, n$), then $g(\sum_i A_i) \geq \sum_i g(A_i)$ for every operator convex function $g(t)$ on $[0, \infty)$ with $g(0) \leq 0$; in particular, $(\sum_i A_i) \log(\sum_i A_i) \geq \sum_i A_i \log A_i$ if each A_i is invertible. Let $A, B \geq 0$ and A be invertible. Then we will observe that the Fréchet derivative $Dg(sA)(B)$ is increasing on $0 < s < \infty$ for every operator convex function $g(t)$ on $(0, \infty)$ if and only if $AB + BA \geq 0$.

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* Corresponding author.

E-mail addresses: uchiama@riko.shimane-u.ac.jp (M. Uchiyama), uchiama@sci.kj.yamagata-u.ac.jp (A. Uchiyama), mariko@nms.ac.jp (M. Giga).

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1. Introduction

Let I be an interval of the real axis and $f(t)$ a real continuous function defined on I . For a bounded Hermitian operator (or matrix) A on a Hilbert space whose spectrum is in I , $f(A)$ stands for the ordinary functional calculus. f is called an *operator monotone* (or *operator decreasing*) function on I if $f(A) \leq f(B)$ (or $f(A) \geq f(B)$) whenever $A \leq B$. It is evident that if $f(t)$ is operator monotone in the interior of I and continuous on I , then $f(t)$ is operator monotone on I itself. It is an essential fact that $f_\lambda(t) := \frac{\lambda t}{\lambda + t}$ is operator monotone on $(-\infty, -\lambda)$ and on $(-\lambda, \infty)$ for each λ . It is also well-known that t^a ($0 < a \leq 1$) is operator monotone on $[0, \infty)$ and so is $\log t$ on $(0, \infty)$. A continuous function g defined on I is called an *operator convex function* on I if $g(sA + (1-s)B) \leq sg(A) + (1-s)g(B)$ for every $0 < s < 1$ and for every pair of bounded Hermitian operators A and B whose spectra are both in I . An *operator concave function* is likewise defined. t^a ($1 \leq a \leq 2$) and $t \log t$ are both operator convex on $[0, \infty)$. For further details, we refer the reader to [2,8]. It has been well-known that a non-negative continuous function $f(t)$ on $[0, \infty)$ is operator monotone if and only if $f(t)$ is operator concave. One of the authors [12,15] (cf. [7]) extended this as follows:

A continuous function $f(t)$ defined on an infinite interval (a, ∞) is operator monotone if and only if $f(t)$ is operator concave and $f(\infty) > -\infty$.

Let $h(t)$ be a non-negative concave (not necessarily operator concave) function on $[0, \infty)$. Since $h(t)$ is increasing and $h(t)/t$ is decreasing, $h(t)$ is subadditive, namely $h(a+b) \leq h(a) + h(b)$. In [3] (also see [1,14]) it was shown that

$$\|h(A+B)\| \leq \|h(A) + h(B)\|$$

for every matrices $A, B \geq 0$ and for every unitarily invariant norm $\|\cdot\|$. Moslehian and Najafi [9] have shown that for $A, B \geq 0$, if $AB + BA \geq 0$, then for any operator monotone function $f(t) \geq 0$ on $[0, \infty)$

$$f(A+B) \leq f(A) + f(B).$$

We will show that if $\sum_{i \neq j} A_i A_j \geq 0$ for bounded self-adjoint operators A_i ($i = 1, 2, \dots, n$), then for every operator monotone function $f(t) \geq 0$ on $[0, \infty)$

$$f(A_1 + \dots + A_n) \leq f(A_1) + \dots + f(A_n),$$

and for every operator convex function with $g(0) \leq 0$

$$g(A_1 + \dots + A_n) \geq g(A_1) + \dots + g(A_n).$$

If $h(t)$ is a C^1 -function defined on an open interval, then the matrix function $h(X)$ is Fréchet differentiable and the derivative $Dh(A)(B)$ equals the Gateaux derivative $\frac{d}{dt} h(A + tB)|_{t=0}$. It is known that if $f(t)$ is operator monotone, the Fréchet derivative

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