Linear Algebra and its Applications 465 (2015) 401–411

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Superadditivity and derivative of operator functions

Mitsuru Uchiyama [a,1], Atsushi Uchiyama [b,*], Mariko Giga [c]

[a] *Department of Mathematics, Interdisciplinary Faculty of Science and Engineering, Shimane University, Matsue city, Shimane 690-8504, Japan*
[b] *Department of Mathematical Sciences, Faculty of Science, Yamagata University, Yamagata city, Yamagata 990-8560, Japan*
[c] *Department of Mathematics, Nippon Medical School, Kawasaki city, Kanagawa 221-0063, Japan*

A R T I C L E   I N F O

Article history:
Received 19 September 2013
Accepted 4 September 2014
Available online 14 October 2014
Submitted by P. Semrl

This paper is dedicated to Professor Tsuyoshi Ando

MSC:
47A56
47A63

Keywords:
Matrix order
Fréchet derivative
Operator monotone function
Operator concave function
Operator convex function

A B S T R A C T

We will show that if $\sum_{i\neq j} A_iA_j \geq 0$ for bounded operators $A_i \geq 0$ $(i = 1, 2, \cdots, n)$, then $g(\sum_i A_i) \geq \sum_i g(A_i)$ for every operator convex function $g(t)$ on $[0, \infty)$ with $g(0) \leq 0$; in particular, $(\sum_i A_i)\log(\sum_i A_i) \geq \sum_i A_i \log A_i$ if each A_i is invertible. Let $A, B \geq 0$ and A be invertible. Then we will observe that the Fréchet derivative $Dg(sA)(B)$ is increasing on $0 < s < \infty$ for every operator convex function $g(t)$ on $(0, \infty)$ if and only if $AB + BA \geq 0$.

* Corresponding author.
E-mail addresses: uchiyama@riko.shimane-u.ac.jp (M. Uchiyama), uchiyama@sci.kj.yamagata-u.ac.jp (A. Uchiyama), mariko@nms.ac.jp (M. Giga).

[1] The author was supported in part by (JSPS) KAKENHI 25400116.

http://dx.doi.org/10.1016/j.laa.2014.09.006
0024-3795/© 2014 Elsevier Inc. All rights reserved.

1. Introduction

Let I be an interval of the real axis and $f(t)$ a real continuous function defined on I. For a bounded Hermitian operator (or matrix) A on a Hilbert space whose spectrum is in I, $f(A)$ stands for the ordinary functional calculus. f is called an *operator monotone* (or *operator decreasing*) function on I if $f(A) \leqq f(B)$ (or $f(A) \geqq f(B)$) whenever $A \leqq B$. It is evident that if $f(t)$ is operator monotone in the interior of I and continuous on I, then $f(t)$ is operator monotone on I itself. It is an essential fact that $f_\lambda(t) := \frac{\lambda t}{\lambda + t}$ is operator monotone on $(-\infty, -\lambda)$ and on $(-\lambda, \infty)$ for each λ. It is also well-known that t^a $(0 < a \leq 1)$ is operator monotone on $[0, \infty)$ and so is $\log t$ on $(0, \infty)$. A continuous function g defined on I is called an *operator convex function* on I if $g(sA + (1-s)B) \leqq sg(A) + (1-s)g(B)$ for every $0 < s < 1$ and for every pair of bounded Hermitian operators A and B whose spectra are both in I. An *operator concave function* is likewise defined. t^a $(1 \leq a \leq 2)$ and $t \log t$ are both operator convex on $[0, \infty)$. For further details, we refer the reader to [2,8]. It has been well-known that a non-negative continuous function $f(t)$ on $[0, \infty)$ is operator monotone if and only if $f(t)$ is operator concave. One of the authors [12,15] (cf. [7]) extended this as follows:
A continuous function $f(t)$ defined on an infinite interval (a, ∞) is operator monotone if and only if $f(t)$ is operator concave and $f(\infty) > -\infty$.

Let $h(t)$ be a non-negative concave (not necessarily operator concave) function on $[0, \infty)$. Since $h(t)$ is increasing and $h(t)/t$ is decreasing, $h(t)$ is subadditive, namely $h(a + b) \leq h(a) + h(b)$. In [3] (also see [1,14]) it was shown that

$$\|h(A + B)\| \leq \|h(A) + h(B)\|$$

for every matrices $A, B \geq 0$ and for every unitarily invariant norm $\| \ \|$. Moslehian and Najafi [9] have shown that for $A, B \geq 0$, if $AB + BA \geq 0$, then for any operator monotone function $f(t) \geq 0$ on $[0, \infty)$

$$f(A + B) \leq f(A) + f(B).$$

We will show that if $\sum_{i \neq j} A_i A_j \geq 0$ for bounded self-adjoint operators A_i $(i = 1, 2, \cdots, n)$, then for every operator monotone function $f(t) \geq 0$ on $[0, \infty)$

$$f(A_1 + \cdots + A_n) \leq f(A_1) + \cdots + f(A_n),$$

and for every operator convex function with $g(0) \leq 0$

$$g(A_1 + \cdots + A_n) \geq g(A_1) + \cdots + g(A_n).$$

If $h(t)$ is a C^1-function defined on an open interval, then the matrix function $h(X)$ is Fréchet differentiable and the derivative $Dh(A)(B)$ equals the Gateaux derivative $\frac{d}{dt} h(A + tB)|_{t=0}$. It is known that if $f(t)$ is operator monotone, the Fréchet derivative