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Eigenvalues and energy in threshold graphs

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ABSTRACT

Assuming a uniform random model of selecting creation sequences, we show that almost every connected threshold graph has more negative than positive eigenvalues. We show that no threshold graphs have eigenvalues in $(-1, 0)$. Sequences of equienergetic graphs are given including the striking result that for all $n \geq 3$, there exist $n-1$ threshold graphs of order n^2, pairwise noncospectral, each equienergetic to K_{n^2}. For all $n > 8$, we find an n-vertex hyperenergetic threshold graph.

1. Introduction

Given $G = (V, E)$, an undirected graph with vertices $V = (v_1, \ldots, v_n)$ and edge set E, its *adjacency matrix* $A(G) = [a_{ij}]$ is the $n \times n$ matrix of zeros and ones for which $a_{ij} = 1$ if and only if v_i is adjacent to v_j (that is, there is an edge between v_i and v_j).

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A value λ is an *eigenvalue* of G if $\det(A - \lambda I_n) = 0$, and since A is real symmetric, its eigenvalues are real.

This paper is concerned with *threshold graphs*, introduced by Chvátal and Hammer [3] and Henderson and Zalcstein [8] in 1977. They are an important class of graphs because of their numerous applications [11]. Recall that a vertex is *isolated* if it has no neighbors, and is *dominating* if it is adjacent to all other vertices. One way to characterize threshold graphs is through an iterative process which starts with an isolated vertex, and where, at each step, either a new isolated vertex is added, or a dominating vertex is added. We represent a threshold graph G on n vertices using a binary string $b_1 \dots b_n$. Here $b_i = 0$ if vertex v_i was added as an isolated vertex, and $b_i = 1$ if v_i was added as a dominating vertex. We call our representation a *creation sequence*, and always take b_1 to be zero.[1] If $n \geq 2$, G is connected if and only if $b_n = 1$. In constructing an adjacency matrix, we order the vertices in the same way they are given in their creation sequence.

Let $\lambda_{\min,n}$ denote the minimum eigenvalue among all threshold graphs of order n. In [9] the authors showed that for $n \geq 3$, the graph $0^k 1^j$ has eigenvalue $\lambda_{\min,n}$, where $k = n - \lfloor \frac{n}{3} \rfloor$ and $j = \lfloor \frac{n}{3} \rfloor$. If $n \equiv 2 \bmod 3$ the graph $0^{k-1}1^{j+1}$ also has eigenvalue $\lambda_{\min,n}$.

Given a graph G, we let $n_+(G)$ and $n_-(G)$ denote respectively the number of positive and negative eigenvalues of G, and $n_0(G)$ and $n_{-1}(G)$ denote the multiplicities of 0 and -1. The triple $(n_+(G), n_0(G), n_-(G))$ is called the *inertia* of G. Bapat showed [1] that for connected threshold graphs G, $n_0(G)$ is the number of 0's in the creation sequence G that are preceded by a 0, and that $n_-(G)$ is the number of 1's in the creation sequence.

In the present paper we investigate further the eigenvalue distribution of threshold graphs. In particular, in Section 2 we show that among connected threshold graphs, the probability of $n_-(G) > n_+(G)$ converges to 1, as $n \to \infty$ assuming a uniform random model of selecting creation sequences. Expressed differently, for *almost every* connected threshold graph G, $n_-(G) > n_+(G)$.

The eigenvalues -1 and 0 in threshold graphs are counterexamples to properties that hold for other numbers. For example, all eigenvalues except -1 and 0 are *main*, meaning that the entries in the associated eigenvector do not sum to zero (see [13, Theorem 7.5]). With the exception of -1 and 0, all eigenvalues of threshold graphs are simple [9]. Their multiplicities can be derived from the creation sequence, as discussed in Theorem 1 and Theorem 6.

It is curious and somewhat unexpected that no threshold graph has an eigenvalue in the open interval $(-1, 0)$. This is shown in Section 4. The technique used is a diagonalization algorithm presented in [9] and reviewed in Section 3. In Section 5 we derive an alternate proof for a formula for $n_{-1}(G)$.

If G is a graph having eigenvalues $\lambda_1, \dots, \lambda_n$ its *energy*, denoted $E(G)$, is defined [10] to be $\sum_{i=1}^{n} |\lambda_i|$. Two nonisomorphic graphs with the same energy are called *equienergetic*. Two graphs with the same spectrum are *cospectral*. Since cospectral graphs are obviously

[1] While some authors omit b_1, this makes it difficult to represent K_1.