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Some operator convex functions of several variables

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ABSTRACT

We obtain operator concavity (convexity) of some functions of two or three variables by using perspectives of regular operator mappings of one or several variables. As an application, we obtain, for $0 < p < 1$, concavity, respectively convexity, of the Fréchet differential mapping associated with the functions $t \to t^{1+p}$ and $t \to t^{1-p}$.

1. Introduction and preliminaries

We study convexity or concavity of certain operator mappings. Some of them may be expressed by the functional calculus for functions of several variables while others are of a more general nature.

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1.1. The functional calculus

Let $\mathcal{H}$ denote an n-dimensional Hilbert space. The space $B(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ is itself a Hilbert space with inner product given by $(A,B) = \mathrm{Tr}(B^*A)$ for $A,B \in B(\mathcal{H})$.

Definition 1.1. Let $f: I_1 \times \cdots \times I_k \to \mathbb{R}$ be a function defined in a product of real intervals, and let $X_1,\dots,X_k$ be commuting operators on $\mathcal{H}$ with spectra $\sigma(X_i) \subseteq I_i$ for $i = 1,\cdots,k$. We say that the k-tuple $(X_1,\dots,X_k)$ is in the domain of f. Consider the spectral resolution

$$X_m = \sum_{i_m=1}^{n_m} \lambda_{i_m}(m) P_{i_m}$$

where $\lambda_1(m),\dots,\lambda_{n_m}(m)$ for $m = 1,\dots,k$ are the eigenvalues of X_m. The functional calculus is defined by setting

$$f(X_1,\dots,X_k) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} f\big(\lambda_{i_1}(1),\dots,\lambda_{i_k}(k)\big) P_{i_1}(1)\cdots P_{i_k}(k)$$

which makes sense since $\lambda_{i_m}(m) \in I_m$ for $i_m = 1,\dots,n_m$ and $m = 1,\dots,k$.

Since the operators $X_1,\dots,X_k$ in the above definition are commuting all of the spectral projections $P_{i_m}(m)$ do also commute. The functional calculus therefore defines $f(X_1,\dots,X_k)$ as a self-adjoint operator on $\mathcal{H}$. Notice that if the tuples $(X_1,\dots,X_k)$ and $(Y_1,\dots,Y_k)$ are in the domain of f then so is the tuple $(\lambda X_1 + (1-\lambda)Y_1,\dots,\lambda X_k + (1-\lambda)Y_k)$ for $\lambda \in [0,1]$.

In order to study convexity properties of the functional calculus it is convenient to consider commuting C^*-subalgebras $\mathcal{A}_1,\dots,\mathcal{A}_k$ of $B(\mathcal{H})$ and require that $X_m \in \mathcal{A}_m$ for $m = 1,\dots,k$. For more details on the functional calculus the reader may refer to [12,7,8].

The restriction of the functional calculus by f to k-tuples of operators $(X_1,\dots,X_k) \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_k$ in the domain of f is said to be convex if

$$\begin{aligned} &f\big(\lambda X_1 + (1-\lambda)Y_1,\dots,\lambda X_k + (1-\lambda)Y_k\big) \\ &\quad \le \lambda f(X_1,\dots,X_k) + (1-\lambda) f(Y_1,\dots,Y_k) \end{aligned}$$

for $\lambda \in [0,1]$.

Definition 1.2. Let $f: I_1 \times \cdots \times I_k \to \mathbb{R}$ be a function defined in a product of real intervals. We say that f is matrix convex of order n if the restriction of the functional calculus by f to operators $(X_1,\dots,X_k) \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_k$ in the domain of f is convex for arbitrary commuting C^*-subalgebras $\mathcal{A}_1,\dots,\mathcal{A}_k$ of $B(\mathcal{H})$.