Linear Algebra and its Applications 463 (2014) 33–44

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The extreme eigenvalues and maximum degree of k-connected irregular graphs

Xiaodan Chen [a,b,*], Yaoping Hou [a]

[a] Department of Mathematics, Hunan Normal University, Changsha 410081, Hunan, PR China

[b] College of Mathematics and Information Science, Guangxi University, Nanning 530004, Guangxi, PR China

ARTICLE INFO

Article history:
Received 17 March 2014
Accepted 25 August 2014
Available online 16 September 2014
Submitted by D. Stevanović

MSC:
05C50
15A18

Keywords:
Irregular graph
k-Connected graph
The largest eigenvalue
The smallest eigenvalue
Maximum degree

ABSTRACT

Let $\lambda_1(G)$ be the largest eigenvalue and $\lambda_n(G)$ be the smallest eigenvalue of a k-connected irregular graph G with n vertices, m edges and maximum degree Δ. In this paper, we prove that

$$\Delta - \lambda_1(G) > \frac{(n\Delta - 2m)k^2}{(n\Delta - 2m)[n^2 - 2(n-k)] + nk^2}.$$

This inequality improves previous results of several authors and implies two lower bounds on $\Delta + \lambda_n(G)$ which also refine some known bounds. Another lower bound on $\Delta - \lambda_1(G)$ for a connected irregular graph G is given as well.

1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be such a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$, where $|E(G)| = m$.

* Corresponding author at: College of Mathematics and Information Science, Guangxi University, Nanning 530004, Guangxi, PR China.

E-mail addresses: x.d.chen@live.cn (X. Chen), yphou@hunnu.edu.cn (Y. Hou).

http://dx.doi.org/10.1016/j.laa.2014.08.020
0024-3795/

For $i = 1, 2, \cdots, n$, let $N_G(v_i)$ denote the set of vertices adjacent to the vertex v_i in G and let $d_G(v_i) = |N_G(v_i)|$ be the degree of v_i. We also denote by $\Delta(G)$ and $\delta(G)$ the maximum degree and the minimum degree of vertices in G, respectively. If $\Delta(G) = \delta(G)$, then G is regular. The distance between any two vertices v_i and v_j in G is the number of edges in a shortest path connecting v_i and v_j. The diameter $D(G)$ of G is the maximum distance over all pairs of vertices. The (vertex) connectivity $\kappa(G)$ of G is the minimum number of vertices whose removal disconnects G or reduces it to a single vertex. For an integer $k \geq 1$, G is called k-connected if $\kappa(G) \geq k$.

The largest eigenvalue, or spectral radius, of G, denoted by $\lambda_1(G)$, is the largest eigenvalue of its adjacency matrix $A(G)$. If G is connected, then $A(G)$ is irreducible and by Perron–Frobenius Theorem (see e.g., [7]), $\lambda_1(G)$ is simple and has a unique positive unit eigenvector. Moreover, it is well-known that $\lambda_1(G) \leq \Delta(G)$ with equality if and only if G is regular. Thus a natural question arises:

How small $\Delta(G) - \lambda_1(G)$ can be when G is irregular?

It is worth mentioning that $\Delta(G) - \lambda_1(G)$ is considered as a measure for the irregularity of a graph G (see [5, p. 242]).

A number of estimates on $\Delta(G) - \lambda_1(G)$ for a connected irregular graph G have been obtained by several authors in recent years. Set $\Delta(G) = \Delta$, $\delta(G) = \delta$ and $D(G) = D$, for convenience. Stevanović [15] first proved that

$$\Delta - \lambda_1(G) > \frac{1}{2n(n\Delta - 1)\Delta^2}.$$

Later in [18], by studying the ratio of a maximal eigenvector, Zhang obtained a finer bound:

$$\Delta - \lambda_1(G) > \frac{\Delta + \delta - 2\sqrt{\Delta\delta}}{nD\Delta} \geq \frac{2\Delta - 1 - 2\sqrt{\Delta(\Delta - 1)}}{n(n-1)\Delta}.$$

Liu, Shen and Wang [8] established an asymptotically best possible bound (up to a constant factor) by investigating the diameter property of so-called λ_1-extremal graphs,

$$\Delta - \lambda_1(G) > \frac{1}{n(D+1)} \geq \frac{\Delta + 1}{n(3n + 2\Delta - 4)}.$$

On the other hand, by introducing other graph parameters, Cioabă, Gregory and Nikiforov [2] showed that

$$\Delta - \lambda_1(G) > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)} \left(\geq \frac{1}{n(D+1)}\right).$$

Further, Cioabă [3] deduced a beautiful bound as follows: