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Log-convexity of Aigner–Catalan–Riordan numbers

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A B S T R A C T

Let $T = [t_{n,k}]_{n,k\geq 0}$ be an infinite lower triangular matrix defined by

$$t_{0,0} = 1, \qquad t_{n+1,0} = \sum_{j=0}^{n} z_j t_{n,j}, \qquad t_{n+1,k+1} = \sum_{j=k}^{n} a_{j,k} t_{n,j}$$

for $n, k \geq 0$, where all $z_j, a_{j,k}$ are nonnegative and $a_{j,k} = 0$ unless $j \geq k \geq 0$. We show that the sequence $(t_{n,0})_{n\geq 0}$ is log-convex if the coefficient matrix $[\zeta, A]$ is TP_2, where $\zeta = [z_0, z_1, z_2, \ldots]'$ and $A = [a_{i,j}]_{i,j\geq 0}$. This gives a unified proof of the log-convexity of many well-known combinatorial sequences, including the Catalan numbers, the Motzkin numbers, the central binomial coefficients, the Schröder numbers, the Bell numbers, and so on.

1. Introduction

Let $(a_n)_{n\geq 0}$ be a sequence of nonnegative numbers. We say that the sequence is *log-convex* (*log-concave*, resp.) if $a_m a_{n+1} \geq a_{m+1} a_n$ ($a_m a_{n+1} \leq a_{m+1} a_n$, resp.) for $0 \leq m < n$. Log-convex and log-concave sequences arise often in combinatorics. An effect

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approach to attack the log-concavity and log-convexity problems comes from the theory of total positivity. We say that an infinite matrix of nonnegative numbers is TP_2 if its minors of order 2 are all nonnegative. Let $(a_n)_{n\geq 0}$ be an infinite sequence of nonnegative numbers and with no internal zeros. Then it is log-concave if and only if its Toeplitz matrix $[a_{i-j}]_{i,j\geq 0}$ is TP_2, and it is log-convex if and only if its Hankel matrix $[a_{i+j}]_{i,j\geq 0}$ is TP_2. We refer the reader to [5–8,13,19,22,23] for total positivity and log-concavity problems. In the present paper we use the concept of total positivity to establish a criterion for the log-convexity of the 0th column $(t_{n,0})_{n\geq 0}$ of an infinite lower triangular matrix

$$T = [t_{n,k}]_{n,k\geq 0} = \begin{bmatrix} t_{0,0} & & & \\ t_{1,0} & t_{1,1} & & \\ t_{2,0} & t_{2,1} & t_{2,2} & \\ & \cdots & & \ddots \end{bmatrix}$$

defined by the recursive system

$$t_{0,0} = 1, \qquad t_{n+1,0} = \sum_{j=0}^{n} z_j t_{n,j}, \qquad t_{n+1,k+1} = \sum_{j=k}^{n} a_{j,k} t_{n,j} \tag{1.1}$$

for $n, k \geq 0$, where all $z_j, a_{j,k}$ are nonnegative and $a_{j,k} = 0$ unless $j \geq k \geq 0$.

The triangles defined by (1.1) are ubiquitous in combinatorics. A basic example is the famous Pascal triangle. We will consider two classes of particular interesting generalizations of the Pascal triangle. The first class of triangles $[c_{n,k}]_{n,k\geq 0}$, introduced by Aigner [2–4], is defined by

$$\begin{aligned} &c_{0,0} = 1, \qquad c_{0,k} = 0 \quad (k > 0), \\ &c_{n+1,k} = c_{n,k-1} + s_k c_{n,k} + t_{k+1} c_{n,k+1} \quad (n, k \geq 0). \end{aligned}$$

The elements $c_{n,0}$ are called the *Catalan-like numbers* corresponding to (σ, τ), where

$$\sigma = (s_0, s_1, s_2, \ldots), \qquad \tau = (t_1, t_2, t_3, \ldots).$$

The Catalan-like numbers unify many well-known counting coefficients. For example, $c_{n,0}$ are

(1) the Catalan numbers C_n corresponding to $\sigma = (1, 2, 2, \ldots)$ and $\tau = (1, 1, 1, \ldots)$;
(2) the Motzkin numbers M_n corresponding to $\sigma = \tau = (1, 1, 1, \ldots)$;
(3) the central binomial coefficients $\binom{2n}{n}$ corresponding to $\sigma = (2, 2, 2, \ldots)$ and $\tau = (2, 1, 1, \ldots)$;
(4) the Schröder numbers S_n corresponding to $\sigma = (2, 3, 3, \ldots)$ and $\tau = (2, 2, 2, \ldots)$;
(5) the Bell numbers B_n corresponding to $\sigma = \tau = (1, 2, 3, 4, \ldots)$.