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The inertia sets of graphs with a 2-separation

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A R T I C L E I N F O

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For a graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$, let $\mathcal{S}(G)$ be the set of all symmetric real $n \times n$ matrices $A = [a_{i,j}]$ with $a_{i,j} \neq 0$, $i \neq j$ if and only if $ij \in E$. The inertia set of a graph G is the set of all possible inertias of matrices in $\mathcal{S}(G)$. In this paper we give a formula that expresses the inertia set of a graph with a 2-separation in terms of the inertia sets of subgraphs of G. This formula follows from an extension to arbitrary fields with characteristic not equal to two.

1. Introduction

The *inertia* of a symmetric real $n \times n$ matrix A is the triple (p, q, r) with p the number of positive eigenvalues, q the number of negative eigenvalues, and r the nullity, respectively, where we take the multiplicities of the eigenvalues into account. Clearly, $n = p + q + r$. The *partial inertia* of A, denoted $\operatorname{pin}(A)$, is the pair (p, q).

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For a graph $G = (V, E)$ with $V = \{1, 2, \dots, n\}$ and a field $\mathbb{F}$, let $\mathcal{S}(G; \mathbb{F})$ be the set of all symmetric $n \times n$ matrices $A = [a_{i,j}]$ with entries in $\mathbb{F}$ and with

1. $a_{i,j} = 0$ if $i \neq j$ and i and j are not adjacent,
2. $a_{i,j} \neq 0$ if $i \neq j$ and there is exactly one edge between i and j,
3. $a_{i,j} \in \mathbb{F}$ if $i \neq j$ and there are multiple edges between i and j,
4. $a_{i,i} \in \mathbb{F}$ for $i \in V$.

We will denote $\mathcal{S}(G; \mathbb{R})$ by $\mathcal{S}(G)$. The inverse inertia problem for G asks what inertias are attained by matrices in $\mathcal{S}(G)$. This problem has been first extensively studied by Barrett et al. [2], and extends the problem of finding the minimum rank of a graph; see [4] for a survey of the minimum rank problem of graphs. The inertia set of a graph G is defined as the set of partial inertias of all matrices in $\mathcal{S}(G)$; that is,

$$\mathcal{I}(G) = \{\operatorname{pin}(A) \mid A \in \mathcal{S}(G)\}.$$

The minimum rank of G, denoted $\operatorname{mr}(G)$, is the smallest rank over all matrices in $\mathcal{S}(G)$. Clearly, $\operatorname{mr}(G) = \min\{p + q \mid (p, q) \in \mathcal{I}(G)\}$.

A *separation* of a graph $G = (V, E)$ is a pair of subgraphs (G_1, G_2) such that $V(G_1) \cup V(G_2) = V$, $E(G_1) \cup E(G_2) = E$, and $E(G_1) \cap E(G_2) = \emptyset$. The *order* of a separation is $|V(G_1) \cap V(G_2)|$. A k-separation is a separation of order k. For a set $\mathcal{I}$ of pairs of nonnegative integers and a nonnegative integer n, define $[\mathcal{I}]_n = \{(p, q) \mid (p, q) \in \mathcal{I}, p+q \leq n\}$. For sets $\mathcal{I}_1$ and $\mathcal{I}_2$ of pairs of nonnegative integers, define $\mathcal{I}_1 + \mathcal{I}_2 = \{(p_1+p_2, q_1+q_2) \mid (p_1, q_1) \in \mathcal{I}_1, (p_2, q_2) \in \mathcal{I}_2\}$.

Let (G_1, G_2) be a 1-separation of a graph $G = (V, E)$ with $V = \{1, 2, \dots, n\}$ and let r be the common vertex of G_1 and G_2. Barrett et al. [2] showed among other things that

$$\mathcal{I}(G) = \big[\mathcal{I}(G_1) + \mathcal{I}(G_2)\big]_n \cup \big[\mathcal{I}(G_1 - r) + \mathcal{I}(G_2 - r) + \{(1, 1)\}\big]_n.$$

In [6], we gave a formula which allows to compute the minimum rank of a graph G with a 2-separation (G_1, G_2) using the minimum ranks of certain small variations of the subgraphs G_1 and G_2. In this paper, we prove that a similar formula holds for the inertia set of a graph G with a 2-separation (G_1, G_2). That is, we prove the following theorem.

Theorem 1. *Let (G_1, G_2) be a 2-separation of a graph G with n vertices, let H_1 and H_2 be obtained from G_1 and G_2, respectively, by adding an edge between the vertices of $R = \{r_1, r_2\} = V(G_1) \cap V(G_2)$, and let $\overline{G_1}$ and $\overline{G_2}$ be obtained from G_1 and G_2, respectively, by identifying r_1 and r_2. Then*

$$\begin{aligned}\mathcal{I}(G) = {} & \big[\mathcal{I}(G_1) + \mathcal{I}(G_2)\big]_n \\ & \cup \big[\mathcal{I}(H_1) + \mathcal{I}(H_2)\big]_n \\ & \cup \big[\mathcal{I}(\overline{G_1}) + \mathcal{I}(\overline{G_2}) + \{(1, 1)\}\big]_n\end{aligned}$$