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A tree-based approach to joint spectral radius determination

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ABSTRACT

We suggest a novel method to determine the joint spectral radius of finite sets of matrices by validating the finiteness property. It is based on finding a certain finite tree with nodes representing sets of matrix products. Our approach accounts for cases where one or several matrix products satisfy the finiteness property. Moreover, is potentially functional even for reducible sets of matrices.

1. Introduction

The concept of the *joint spectral radius (JSR)* has gained some attention in recent years. As a generalization of the standard spectral radius, this characteristic of a set of matrices plays an important role in various fields of modern mathematics, see e.g. the monograph [18]. First introduced by Rota and Strang in 1960 [23], the JSR was almost forgotten, and then rediscovered in 1992 by Daubechies and Lagarias [7] in the context of the analysis of refineable functions. In general, the JSR is not exactly computable [3], and even its approximation is NP-hard in some sense [2]. In practice, the situation is as follows: Beyond the analysis of special cases, the few known algorithms for a potential

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exact evaluation are based on establishing the *finiteness property (FP)*, as introduced below, for a given problem. This approach cannot claim universality because families of matrices exist[1] that do not exhibit the FP. However, to the best of our knowledge, all problems coming from applications have permitted a treatment via FP so far.

In [10,5,14,13,21,12,20], the FP is established by the construction of a special norm with certain extremal properties. The approach suggested here also aims at the FP but is different in the way how it is verified. Some pros and cons will be discussed in the final section of this paper.

The idea of a graph-theoretical analysis has proven to be useful before. In [11], a branch and bound algorithm on the tree of all matrix products is used for approximating the JSR. This method yields arbitrarily small enclosing intervals but, except for some very special cases, cannot be used for an exact determination. In [16], the range of C^1-parameters of the four-point scheme is determined explicitly by considering certain infinite paths in the tree of all matrix products. Though we adapted some ideas from the latter work, the trees to be considered in the following are different from those in [11] and [16]: Their knots represent *sets of matrices* instead of single matrix products. This crucial idea potentially reduces the analysis of infinite sets of products to a study of *finite* subtrees. In particular, this aspect facilitates automated verification of the FP by computer programs.

After introducing some notation, our main results are presented in Section 3 and then proven in the subsequent section. Section 5 illustrates our new method by some model problems but does not comprise numerical tests since the focus of this work is on theoretical aspects. In particular, we demonstrate the capabilities of our method by settling two problems raised in [12]. Some concluding remarks can be found in Section 6.

2. Setup

We consider a finite set $\mathcal{A} = \{A_1, ..., A_m\}$ of matrices in $\mathbb{C}^{d\times d}$. To deal with products of its elements, we introduce the sets

$$\mathcal{I}_0 := \{\emptyset\}, \qquad \mathcal{I}_k := \{1, \dots, m\}^k, \qquad \mathcal{I} := \bigcup_{k\in\mathbb{N}_0} \mathcal{I}_k,$$

of *completely positive index vectors* of length $k \in \mathbb{N}_0$ and arbitrary length, respectively. By contrast, an *index vector* may contain also negative entries, whose special meaning will be explained in the next section.

For $k \in \mathbb{N}$, we define the matrix product

$$A_I := A_{i_k} \cdots A_{i_1}, \quad I = [i_1, \dots, i_k] \in \mathcal{I}_k. \tag{1}$$

[1] Non-constructive proofs of this fact can be found in [1,4,19], while an explicit counterexample is given in [17].