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Maps on classes of Hilbert space operators preserving measure of commutativity

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A R T I C L E I N F O

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A B S T R A C T

In this paper first we give a partial answer to a question of L. Molnár and W. Timmermann. Namely, we will describe those linear (not necessarily bijective) transformations on the set of self-adjoint matrices which preserve a unitarily invariant norm of the commutator. After that we will characterize those (not necessarily linear or bijective) maps on the set of self-adjoint rank-one projections acting on a two-dimensional complex Hilbert space which leave the latter quantity invariant. Finally, this result will be applied in order to obtain a description of such bijective preservers on the unitary group and on the set of density operators.

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1. Introduction and statement of the results

The relation of commutativity appears in most fields of mathematics and therefore the investigation of commutativity preserving transformations is a relevant problem. Such preservers on a certain class of operators are extremely important because they are connected to quantum mechanics. Namely, in the mathematical formalism of quantum mechanics a complex (and in most cases separable) Hilbert space can be associated to every quantum system. The so called observables correspond to self-adjoint operators, the pure states or rays are identified with self-adjoint rank-one projections, the mixed states are represented by density operators. The commutativity of these representing operators has a certain physical meaning.

The structure of mappings that preserve commutativity (usually in both directions) was investigated in many papers for different classes of operators, see for instance: [4,9, 11–13,15]. For several classes of normal operators it turned out that such bijections send each element – up to unitary or antiunitary equivalence – into a certain bounded Borel function of it. However, it is important to note that usually these results are valid only for Hilbert spaces with at least three dimensions. For example, in a two-dimensional space two self-adjoint operators commute if and only if they are linearly dependent or there exists a real-linear combination of them which equals the identity operator. Therefore in two dimensions many transformations exist on the set of self-adjoint operators which preserve commutativity in both directions.

However, if we pose a stronger condition on our transformation rather than simply the preservation of commutativity in both directions, we shall obtain more regular forms. One natural possibility is to consider the operator norm of the commutator and investigate such transformations that preserve this quantity. Concerning this kind of preservers, recently, L. Molnár and W. Timmermann proved a theorem which is stated below, but before that we give some auxiliary definitions. Let $\mathcal{H}$ denote a complex and at least two-dimensional Hilbert space. The symbols $\mathcal{B}(\mathcal{H}), \mathcal{B}_s(\mathcal{H}), \mathcal{P}_1(\mathcal{H}), \mathcal{U}(\mathcal{H}), \mathcal{S}(\mathcal{H})$ will denote the set of bounded linear operators, bounded self-adjoint operators, self-adjoint rank-one projections, unitary operators and density operators acting on $\mathcal{H}$, respectively. We note that a positive operator A is said to be a density operator if $\operatorname{Tr} A = 1$ where Tr stands for the trace. The operator norm of an element $A \in \mathcal{B}(\mathcal{H})$ will be denoted by $\|A\|$, and the vector norm of a vector $h \in \mathcal{H}$ by $\|h\|$. A norm $|||\cdot|||$ on $\mathcal{B}(\mathcal{H})$ is called unitarily invariant if $|||UAV||| = |||A|||$ is satisfied whenever $A \in \mathcal{B}(\mathcal{H})$ and $U, V \in \mathcal{U}(\mathcal{H})$. The reader can find a characterization of all unitarily invariant norms on matrices in [2, Section IV.2], which will be used many times throughout the paper. The commutator of two operators A, B is the operator $AB - BA$ which is usually denoted by $[A, B]$. The previously mentioned result of Molnár and Timmermann reads as follows.

Theorem. *(See L. Molnár and W. Timmermann [10].) Let $\mathcal{H}$ be separable with* $\dim \mathcal{H} > 2$. *Assume $\phi: \mathcal{B}_s(\mathcal{H}) \to \mathcal{B}_s(\mathcal{H})$ is a bijection such that*