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Companion matrix patterns ☆

B. Eastman [a], I.-J. Kim [b], B.L. Shader [c], K.N. Vander Meulen [a,*]

[a] *Department of Mathematics, Redeemer University College, Ancaster, ON, L9K 1J4, Canada*
[b] *Department of Mathematics and Statistics, Minnesota State University, Mankato, MN 56001, USA*
[c] *Department of Mathematics, University of Wyoming, Laramie, WY 82071-3036, USA*

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ABSTRACT

Companion matrices, especially the Frobenius companion matrices, are used in algorithms for finding roots of polynomials and are also used to find bounds on eigenvalues of matrices. In 2003, Fiedler introduced a larger class of companion matrices that includes the Frobenius companion matrices. One property of the combinatorial pattern of these companion matrices is that, up to diagonal similarity, they uniquely realize every possible spectrum of a real matrix. We characterize matrix patterns that have this property and consequently introduce more companion matrix patterns. We observe that each Fiedler companion matrix is permutationally similar to a unit Hessenberg matrix. We provide digraph characterizations of the classes of patterns described, and in particular, all sparse companion matrices, noting that there are companion matrices that are not sparse.

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* Corresponding author.
E-mail addresses: beastman@redeemer.ca (B. Eastman), in-jae.kim@mnsu.edu (I.-J. Kim), bshader@uwyo.edu (B.L. Shader), kvanderm@redeemer.ca (K.N. Vander Meulen).

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1. Introduction

The *Frobenius companion matrix* is

$$C_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix}.$$

In some contexts, instead of C_n, authors use a matrix that can be obtained from C_n via transpose, or via a reverse permutation similarity or both. (An $n \times n$ reverse permutation matrix P is a permutation matrix with $P_{i,n-i+1} = 1$ for $1 \leq i \leq n$.) A key property of the Frobenius companion matrix is that the variables entries in C_n provide the coefficients of the characteristic polynomial of C_n:

$$p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n. \tag{1}$$

We define a *companion matrix* to be an $n \times n$ matrix A over $\mathbb{F}[a_1, a_2, \ldots, a_n]$ with $n^2 - n$ entries constant in $\mathbb{F}$ and the remaining entries $-a_1, -a_2, \ldots, -a_n$ such that the characteristic polynomial of A is $p(x)$ in line (1). We say that two companion matrices A and B are *equivalent* if either A or A^T can be obtained from B via a permutation similarity.

In [8], Fiedler introduces some new companion matrices. For $t = 1, \ldots, n-1$, let

$$M_t = \begin{bmatrix} I_{t-1} & O & O \\ O & N_t & O \\ O & O & I_{n-t-1} \end{bmatrix} \quad \text{where } N_t = \begin{bmatrix} -a_t & 1 \\ 1 & 0 \end{bmatrix}$$

and let M_n be a diagonal matrix with main diagonal entries $(1, 1, \ldots, 1, -a_n)$. Fiedler [8] demonstrated that if $(i_1, i_2, \ldots, i_n)$ is any permutation of $\{1, 2, \ldots, n\}$ then the product $M_{i_1} M_{i_2} \cdots M_{i_n}$ is a companion matrix, which we will call a *Fiedler companion matrix*. The Frobenius companion matrix is equivalent to the Fiedler companion matrix $M_1 M_2 \cdots M_n$ via a reverse permutation similarity. Fiedler companion matrices are currently being explored as an alternate computational tool to the Frobenius companion matrices by various authors; see, for example, [1,3,4,12–14].

In this paper, we provide a characterization of all companion matrices, up to equivalence, and demonstrate that there are companion matrices that are not Fiedler companion matrices. For example, the matrices