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Hamiltonian actions on the cone of positive definite matrices ☆

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The semigroup of Hamiltonians acting on the cone of positive definite matrices via linear fractional transformations satisfies the Birkhoff contraction formula for the Thompson metric. In this paper we describe the action of the Hamiltonians lying in the boundary of the semigroup. This involves in particular a construction of linear transformations leaving invariant the cone of positive definite matrices (strictly positive linear mappings) parameterized over all square matrices. Its invertibility and relation to the Lyapunov and Stein operators are investigated in detail. In particular, it is shown that each of these linear transformations commutes with the corresponding Lyapunov operator and contracts the Thompson metric.

1. Introduction

Let Sp_m be the symplectic Lie group of $2m \times 2m$ block matrices $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ such that A^*C, B^*D are Hermitian and $A^*D - C^*B = I$, equivalently

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$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

The symplectic Lie algebra $\mathfrak{sp}_m$ consists of all $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ such that B and C are Hermitian and $D = -A^*$, that is,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 0.$$

The symplectic Lie group Sp_m acts partially via fractional transformations on the space $\mathbb{H}$ of $m \times m$ Hermitian matrices:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} .X = (AX + B)(CX + D)^{-1}, \quad \text{if } (CX + D)^{-1} \text{ exists.} \tag{1.1}$$

A member of Sp_m is called a *compression* or *Hamiltonian* if it carries the open convex cone $\mathbb{P}$ of positive definite matrices into itself under the action of fractional transformation (1.1). The semigroup Γ of Hamiltonians acts as contractions for the Thompson metric

$$d(A, B) = \left\| \log A^{-1/2} B A^{-1/2} \right\|$$

on $\mathbb{P}$, where $\|X\|$ denotes the spectral norm of X, and is more completely described in the next theorem in terms of Birkhoff contraction formula [4] and polar decomposition. See [11,6,7,10,5] for the more general setting of bounded operators on a Hilbert space and of Euclidean Jordan algebras. We note that the classical Birkhoff contraction formula, which characterizes the convergent rate with respect to Hilbert's projective metric, applies to linear mapping that maps a cone into a (possibly different) cone [12,13].

Theorem 1.1. *Let $g \in \Gamma$. Then*

$$L(g) := \sup_{\substack{X,Y \in \mathbb{P} \\ X \neq Y}} \frac{d(g(X), g(Y))}{d(X, Y)} = \tanh\left[\frac{\Delta(g)}{4}\right]$$

where $\Delta(g)$ denotes the diameter of $g(\mathbb{P})$ for the Thompson metric. Moreover,

$$\Gamma = \mathcal{H} \exp \mathcal{W}, \qquad \Gamma^\circ := \operatorname{int}(\Gamma) = \mathcal{H} \exp \mathcal{W}^\circ$$

where

$$\mathcal{W} = \left\{ \begin{bmatrix} M & A \\ B & -M^* \end{bmatrix} : A, B \geq 0 \right\}, \qquad \mathcal{W}^\circ = \left\{ \begin{bmatrix} M & A \\ B & -M^* \end{bmatrix} : A, B > 0 \right\},$$

and $\mathcal{H} = \{ \left[\begin{smallmatrix} D^* & 0 \\ 0 & D^{-1} \end{smallmatrix}\right] : D \in \mathrm{GL}(m) \}$. Furthermore, $L(g) < 1$ if and only if $g \in \Gamma^\circ$.