Linear Algebra and its Applications 460 (2014) 17–50

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Category $\mathcal{O}$ for the Schrödinger algebra

Brendan Dubsky [a], Rencai Lü [b], Volodymyr Mazorchuk [a], Kaiming Zhao [c,d]

[a] *Department of Math., Uppsala University, Box 480, SE-751 06, Uppsala, Sweden*
[b] *Department of Math., Soochow University, Suzhou 215006, Jiangsu, PR China*
[c] *Department of Math., Wilfrid Laurier University, Waterloo, Ontario, N2L 3C5, Canada*
[d] *College of Math. and Information Science, Hebei Normal (Teachers) University, Shijiazhuang 050016, Hebei, PR China*

A R T I C L E   I N F O

Article history:
Received 7 February 2014
Accepted 22 July 2014
Available online 7 August 2014
Submitted by P. Semrl

MSC:
17B10
17B65
17B66
17B68

Keywords:
Lie algebra
Category $\mathcal{O}$
Simple module
Weight module
Indecomposable module
Annihilator

A B S T R A C T

We study category $\mathcal{O}$ for the (centrally extended) Schrödinger algebra. We determine the quivers for all blocks and relations for blocks of nonzero central charge. We also describe the quiver and relations for the finite dimensional part of $\mathcal{O}$. We use this to determine the center of the universal enveloping algebra and annihilators of Verma modules. Finally, we classify primitive ideals of the universal enveloping algebra which intersect the center of the centrally extended Schrödinger algebra trivially.

E-mail addresses: brendan.frisk.dubsky@math.uu.se (B. Dubsky), rencail@amss.ac.cn (R. Lü), mazor@math.uu.se (V. Mazorchuk), kzhao@wlu.ca (K. Zhao).

http://dx.doi.org/10.1016/j.laa.2014.07.030

1. Introduction and description of the results

The Schrödinger Lie group describes symmetries of the free particle Schrödinger equation, see [27]. The corresponding Lie algebra is called the Schrödinger algebra, see [9]. In the 1 + 1-dimensional space–time this algebra is, roughly, a semi-direct product of the simple Lie algebra $\mathfrak{sl}_2$ with its simple 2-dimensional representation (the latter forms an abelian ideal). This Lie algebra admits a universal 1-dimensional central extension which is called the centrally extended Schrödinger algebra or, simply, the Schrödinger algebra, abusing the language.

Some basics of the representation theory of the Schrödinger algebra were studied in [9,10], including description of simple highest weight modules. Recently there appeared a number of papers studying various aspects of the representation theory of the Schrödinger algebra, see [1,19,20,11,30–32]. In particular, [11] classifies all simple modules over the Schrödinger algebra which are weight and have finite dimensional weight spaces.

The present paper started with the observation that the claim of [31, Theorem 1.1(1)] contradicts [27, p. 244] and a natural subsequent attempt to repair the main result of [31] which claims to describe annihilators of Verma modules over the Schrödinger algebra. In the classical situation of simple Lie algebras, a study of annihilators of Verma modules usually follows the study of the BGG category $\mathcal{O}$ and its equivalent realization using Harish–Chandra bimodules. This naturally led us to the problem of understanding category $\mathcal{O}$ for the Schrödinger algebra. This is the main objective of the present paper.

Making a superficial parallel with the theory of affine Lie algebras, it turns out that the representation theory of the Schrödinger algebra splits into two very different cases, namely the case of nonzero central charge and the one of the zero central charge, where by the *central charge* we, as usual, mean the eigenvalue of the (unique up to scalar) central element of the Schrödinger algebra (note that such an eigenvalue is unique for all simple modules). For nonzero central charge our results are complete, whereas for zero central charge we get less information, however, involving much more complicated arguments. Nevertheless, we derive enough properties of $\mathcal{O}$ to be able to describe the center of the universal enveloping algebra of the Schrödinger algebra and annihilators of Verma modules, repairing the main results of [31]. Along the way we also describe the "finite dimensional" part of $\mathcal{O}$ which, in contrast with the classical case, is no longer a semi-simple category. Our description, in particular, implies that the category of finite dimensional modules over the Schrödinger algebra has wild representation type (cf. [22]).

The paper is organized as follows: in Section 2 we collected all necessary preliminaries. Section 3 studies basics on category $\mathcal{O}$ and describes blocks of nonzero central charge. Section 4 studies blocks of zero central charge and the "finite dimensional" part of $\mathcal{O}$. As a technical tool we also introduce a natural graded version of $\mathcal{O}$ (which makes sense only for zero central charge). Section 5 contains several applications, in particular, description of the center of the universal enveloping algebra of the Schrödinger algebra and description of annihilators of Verma modules. In Section 6 we outline the setup to