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On mappings approximately transferring relations in finite-dimensional normed spaces

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ABSTRACT

We discuss the problem of stability of relations preserving property for finite-dimensional normed spaces. Moreover, we obtain similar results for other orthogonality relations. Next, we show a counterexample for infinite-dimensional normed spaces.

1. Introduction

An orthogonality preserving property can be introduced, in the most natural way, for linear mappings between inner product spaces. If X and Y are real or complex inner

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product spaces with the standard orthogonality relation, a linear mapping $f: X \to Y$ which satisfies the condition

$$\forall_{x,y\in X} \quad x \perp y \quad \Rightarrow \quad f(x) \perp f(y)$$

is called *orthogonality preserving* (o.p.). A linear orthogonality preserving mapping has to be a linear similarity (cf. [4]).

Let us now introduce the notion of *approximate orthogonality*. For $\varepsilon \in [0,1)$, we say that vectors u, v are ε-orthogonal ($u \perp^{\varepsilon} v$) whenever $|\langle u|v\rangle| \leqslant \varepsilon\|u\| \cdot \|v\|$. A linear mapping $f: X \to Y$ which satisfies

$$\forall_{x,y\in X} \quad x \perp y \quad \Rightarrow \quad f(x) \perp^{\varepsilon} f(y)$$

is called *ε-approximately orthogonality preserving* (ε-a.o.p.). Linear ε-a.o.p. mappings are, in a sense, approximate similarities (cf. [4]).

Next, we formulate a stability problem: whether for each linear mapping f approximately orthogonality preserving, there exists a linear mapping h preserving orthogonality, which is close to f. The answer is affirmative. If f is ε-a.o.p., then there is h preserving orthogonality such that $\|f - h\| \leqslant (1 - \sqrt{\frac{1-\varepsilon}{1+\varepsilon}})\|f\|$; see [5,10].

In a normed space, one can define various orthogonality relations and one can consider linear mappings (approximately) preserving this relations. For example: *the Birkhoff-orthogonality*

$$x \perp_{\mathrm{B}} y \quad :\Leftrightarrow \quad \forall_{\lambda\in\mathbb{K}} \quad \|x\| \leqslant \|x + \lambda y\|,$$

and *the ε-Birkhoff-orthogonality* (see [7,2,3])

$$x \,{}^{\varepsilon}{\perp}_{\mathrm{B}}\, y \quad :\Leftrightarrow \quad \forall_{\lambda\in\mathbb{K}} \quad (1-\varepsilon)\|x\| \leqslant \|x + \lambda y\|.$$

The above stability problem has been also carried out for $\perp_{\mathrm{B}}$, ${}^{\varepsilon}{\perp}_{\mathrm{B}}$ (see [9]). For real normed space it can be considered *the isosceles orthogonality*:

$$x \perp_{\mathrm{i}} y \quad :\Leftrightarrow \quad \|x + y\| = \|x - y\|,$$

and *the ε-isosceles orthogonality*

$$x \,{}^{\varepsilon}{\perp}_{\mathrm{i}}\, y \quad :\Leftrightarrow \quad \big|\|x + y\| - \|x - y\|\big| \leqslant \varepsilon\big(\|x + y\| + \|x - y\|\big).$$

For $\perp_{\mathrm{i}}$, ${}^{\varepsilon}{\perp}_{\mathrm{i}}$ the stability problem has been solved in [6]. Of course, in an inner product space we have $\perp = \perp_{\mathrm{B}} = \perp_{\mathrm{i}}$. We will consider only finite-dimensional normed spaces, but we will obtain similar results for other orthogonality relations.