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Geometry of standard symmetrized tensors

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The geometric properties of the set of standard (decomposable) symmetrized tensors are studied and some general results are obtained. As an example, the geometry is worked out completely in the case where the group is a dihedral group, and this result is used to give a more conceptual proof of an earlier result. As another example, it is shown that there exists an orbital subspace such that the standard symmetrized tensors in the subspace form a root system isomorphic to a given irreducible root system if and only if the irreducible root system is simply laced.

0. Introduction

Let G be a subgroup of the symmetric group S_n ($n \in \mathbf{N}$) and let V be an inner product space. Orthogonality properties of the set of standard (decomposable) symmetrized tensors in $V^{\otimes n}$ corresponding to G have been studied for more than two decades [12,7,5,3,1,6,11]. The determination of such properties would be facilitated by an understanding

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of the more general geometric properties of this set. We propose a framework for the study of such properties.

The space $V^{\otimes n}$ is an orthogonal direct sum of orbital subspaces, so it is sufficient to study the sets of standard symmetrized tensors in these subspaces. It then follows that it is sufficient to study for each irreducible character χ of G and each subgroup H of G the set Ψ of standard vectors in the coset space $\mathcal{C}_H^\chi$ (see Section 3).

In Section 4 we obtain some general results about the pairs $(\mathcal{C}_H^\chi, \Psi)$. Then in Section 5 we compute all such pairs in the case where G is a dihedral group and use our results to give a more conceptual proof of an earlier result in [6]. Finally, in Section 6 we generalize a result of Torres and Silva [11] by showing that there exists an orbital subspace such that the standard symmetrized tensors in the subspace form a root system isomorphic to a given irreducible root system if and only if the irreducible root system is simply laced.

1. Hermitian form

In this section and the next we review, for the convenience of the reader, some standard (and also some less standard) terminology and results.

Let V be a complex vector space. A function $f : V \times V \to \mathbf{C}$ is a *Hermitian form on* V if for all $u, v, w \in V$ and $\alpha \in \mathbf{C}$ the following hold:

(i) $f(u+v, w) = f(u, w) + f(v, w)$,
(ii) $f(\alpha v, w) = \alpha f(v, w)$,
(iii) $f(v, w) = \overline{f(w, v)}$.

Let f be a Hermitian form on V. It follows from the axioms that f is antilinear in the second argument (meaning $f(u, v+w) = f(u, v) + f(u, w)$ and $f(v, \alpha w) = \bar{\alpha} f(v, w)$ for all $u, v, w \in V$ and $\alpha \in \mathbf{C}$) and that $f(v, v) \in \mathbf{R}$ for all $v \in V$.

The Hermitian form f is *positive semidefinite* if $f(v, v) \geq 0$ for all $v \in V$; it is an *inner product* if it is positive semidefinite and it satisfies the *definite* property: $f(v, v) = 0$ if and only if $v = 0$.

The *kernel* of f is the subspace $\ker f = \{v \in V \mid f(v, w) = 0 \text{ for all } w \in V\}$ of V. Put $\bar{V} = V/\ker f$ and denote by $v \mapsto \bar{v}$ the canonical epimorphism $V \to \bar{V}$. (Context should keep any confusion from arising between this notation and that for complex conjugation.) The function $\bar{f} : \bar{V} \times \bar{V} \to \mathbf{C}$ given by $\bar{f}(\bar{v}, \bar{w}) = f(v, w)$ is a well-defined Hermitian form on $\bar{V}$.

Lemma 1.1. *Let f be a positive semidefinite Hermitian form on V.*

(i) $\ker f = \{v \in V \mid f(v, v) = 0\}$.
(ii) *The function $\bar{f}$ is an inner product on $\bar{V}$.*

Proof. (i) Let $v \in \{v \in V \mid f(v, v) = 0\} =: W$. Then $\|v\| = f(v, v)^{1/2} = 0$, so $|f(v, w)| \leq \|v\|\|w\| = 0$ for all $w \in V$, where we have used the Cauchy–Schwartz inequality (the