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A comparison of different notions of ranks of symmetric tensors

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We introduce various notions of rank for a high order symmetric tensor taking values over the complex numbers, namely: rank, border rank, catalecticant rank, generalized rank, scheme length, border scheme length, extension rank and smoothable rank. We analyze the stratification induced by these ranks. The mutual relations between these stratifications allow us to describe the hierarchy among all the ranks. We show that strict inequalities are possible between rank, border rank, extension rank and catalecticant rank. Moreover we show that scheme length, generalized rank and extension rank coincide.

0. Introduction

Tensor decomposition is one of those issues that often arise in applications (see [32] and the references therein). On the grounds of the numerous analogies with the matrix singular value decomposition (SVD), this multilinear generalization to high order tensors that we are going to consider, is also called the "higher-order singular value

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decomposition (HOSVD)" [24]. HOSVD is a linear algebra method often used to recover geometric or intrinsic information, "hidden" in the tensor data. For a given tensor with a certain structure, it consists of finding the minimal decomposition into indecomposable tensors with the same structure. The best known and studied case is the one of symmetric tensors (see examples in [15,21,22]), i.e. homogeneous polynomials. The minimum number r of indecomposable symmetric tensors $v_i^{\otimes d}$'s (d-th powers of linear forms l_i's) needed to write a given symmetric tensor T of order d (a homogeneous polynomial f of degree d) is called the *rank* $r(T)$ of T (the *rank* $r(f)$ of f):

$$T = \sum_{i=1}^{r} v_i^{\otimes d}; \qquad f = \sum_{i=1}^{r} l_i^d.$$

Observe that the case $d = 2$ (i.e. when T is a symmetric matrix and f is a quadric) coincides with the standard definition of rank of a matrix. In that case, a tensor decomposition of a symmetric matrix of rank r (that can be obtained by SVD computation) allows to write such a matrix as a linear combination of r symmetric matrices of rank 1.

From now on, with an abuse of notation, we will denote by "f" both a symmetric tensor and its associated homogeneous polynomial.

From a geometric point of view, saying that a symmetric tensor f has rank r means that it is a generic element of the r-th secant variety of the Veronese variety in the projective space of homogeneous polynomials of degree d. The order $r_\sigma(f)$ of the smallest secant variety to the Veronese variety containing a given f is called the *border rank* of T and may differ from the rank of f (see Example 2.2).

The first method to decompose a high order symmetric tensor is classically attributed to Sylvester and it works for tensors $f \in V^{\otimes d}$ with $\dim V = 2$ (i.e. for binary forms). Such a method (see [20] for a modern reference) is based on the analysis of the kernels of so-called catalecticant matrices associated to the tensor. This leads to the notion of *catalecticant rank* $r_H(f)$ of a tensor f, which is also called "differential length" in [31, Definition 5.66, p. 198].

Extending the apolarity approach of Sylvester, an algorithm to compute the decomposition and the rank of a symmetric tensor f in any dimension was described in [10]. The main ingredient of that paper is an algebraic characterization of the property of flat extension of a catalecticant matrix. This extension property is not enough to characterize tensors with a given rank, since the underlying scheme associated to the catalecticant matrix extension should be reduced. To get a better insight on this difference, we introduce hereafter the notions of *extension rank* $r_{\mathcal{E}^0}(f)$ and *border extension rank* $r_{\mathcal{E}}(f)$ of f, and we analyze their main properties.

Another approach leading to a different kind of algorithm is proposed in [7] where it is developed for some cases. The idea there is to classify all the possible ranks of the polynomials belonging to certain secant varieties of Veronese varieties in relation to the structure of the embedded non-reduced zero-dimensional schemes whose projective span is contained in that secant variety. In [12], the authors clarify the structure of