Linear Algebra and its Applications 460 (2014) 231–241

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A modification of eigenvalue localization for stochastic matrices

Chaoqian Li, Yaotang Li *

School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan, 650091, PR China

ARTICLE INFO

Article history:
Received 20 February 2014
Accepted 22 July 2014
Available online 14 August 2014
Submitted by R. Brualdi

MSC:
65F15
15A18
15A51

Keywords:
Stochastic matrix
Eigenvalues
Nonnegative matrices

ABSTRACT

A new eigenvalue localization for stochastic matrices is provided, and is used to estimate the moduli of the subdominant eigenvalue. Numerical examples are given to show that our results are better than those in Cvetković, Kostić and Peña (2011) [3].

1. Introduction

An entrywise nonnegative matrix $A = [a_{ij}] \in R^{n\times n}$ is called row stochastic (or simply stochastic) if all its row sums are 1, that is,

$$\sum_{j=1}^{n} a_{ij} = 1, \quad \text{for each } i \in N = \{1, 2, \dots, n\}.$$

* Corresponding author.
E-mail addresses: lichaoqian@ynu.edu.cn (C. Li), liyaotang@ynu.edu.cn (Y. Li).

http://dx.doi.org/10.1016/j.laa.2014.07.032

Obviously, 1 is an eigenvalue of a stochastic matrix with a corresponding eigenvector $e = [1, 1, \dots, 1]^T$. From the Perron–Frobenius Theorem [1], for any eigenvalue λ of A, that is, $\lambda \in \sigma(A)$, we have $|\lambda| \le 1$, so that in fact 1 is a dominant eigenvalue for A [3]. Here we call λ a subdominant eigenvalue of a stochastic matrix A if $1 > |\lambda| > |\eta|$ for all eigenvalues η different from 1 and λ [3,5,6].

Stochastic matrices and eigenvalue localization of stochastic matrices play key roles in many application fields such as Computer Aided Geometric Design [9], Birth–Death Processes [2,4,7,8], and Markov chain [10].

In [3], L.J. Cvetković et al. presented a region including all eigenvalues of a stochastic matrix A different from 1 by refining the Geršgorin circle [11] of A.

Theorem 1. *(See [3, Theorem 3.4].) Let $A = [a_{ij}] \in R^{n\times n}$ be a stochastic matrix, and let s_i be the minimal element among the off-diagonal entries of the i-th column of A, that is, $s_i = \min_{j\neq i} a_{ji}$. Taking $\gamma(A) = \max_{i\in N}(a_{ii} - s_i)$, then for any $\lambda \in \sigma(A)\backslash\{1\}$,*

$$\bigl|\lambda - \gamma(A)\bigr| < r(A) = 1 - \mathit{trace}(A) + (n-1)\gamma(A).$$

Although Theorem 1 provides a circle with the center $\gamma(A)$ and radius equal to $1 - \mathit{trace}(A) + (n-1)\gamma(A)$ to localize the eigenvalue λ, it is not effective in some cases. Consider the following class of matrices

$$SM_0 = \bigl\{A \in R^{n\times n} : A \text{ is stochastic, and } a_{ii} = s_i = 0, \text{ for each } i \in N\bigr\}.$$

Then for any $A \in SM_0$, $\mathit{trace}(A) = 0$ and $\gamma(A) = 0$. Hence by Theorem 1, we have $|\lambda| < 1$ for $\lambda \in \sigma(A)\backslash\{1\}$. This is trivial. It is very interesting how to conquer this drawback.

Note that if A is a stochastic matrix, then A^m is also stochastic for any positive integer m. Therefore, we can apply Theorem 1 to A^m and obtain

$$\bigl|\lambda^m - \gamma\bigl(A^m\bigr)\bigr| \le r\bigl(A^m\bigr) = 1 - \mathit{trace}\bigl(A^m\bigr) + (n-1)\gamma\bigl(A^m\bigr). \tag{1}$$

When A is a positive stochastic matrix, L.J. Cvetković et al. [3] proved that the sequence $\{\gamma(A^m)\}$ converges to 0, and the radii of the corresponding circles $r(A^m) = 1 - \mathit{trace}(A^m) + (n-1)\gamma(A^m)$ also tend to 0.

Theorem 2. *(See [3, Theorem 3.5].) Let $A = [a_{ij}] \in R^{n\times n}$ be a positive stochastic matrix. Then*

$$\lim_{m\to\infty} \gamma\bigl(A^m\bigr) = 0$$

and

$$\lim_{m\to\infty} r\bigl(A^m\bigr) = 0.$$