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The group of automorphisms of a zero-divisor graph based on rank one upper triangular matrices

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Let F_q be a finite field with q elements, $n(\geq 2)$ a positive integer, $\mathrm{Mat}_n(q)$ the set of all $n \times n$ matrices over F_q, $R(n,q)$ the set of all rank one upper triangular matrices in $\mathrm{Mat}(n,q)$. The zero-divisor graph of $\mathrm{Mat}_n(q)$, written as $\Gamma(\mathrm{Mat}_n(q))$, is a directed graph with vertex set all nonzero zero-divisors of $\mathrm{Mat}_n(q)$, and there is a directed edge from a vertex A to a vertex B, written as $A \to B$, if and only if $AB = 0$. In this paper, we determine the automorphisms of an induced subgraph, written as $\Gamma(R(n,q))$, of $\Gamma(\mathrm{Mat}_n(q))$ with vertex set $R(n,q)$. The main theorem of this article proves that a bijective map σ on $R(n,q)$ with $n \geq 3$ is an automorphism of $\Gamma(R(n,q))$ if and only if

$$\sigma(X) = a_X P^{-1}\big[\pi(x_{ij})\big]P, \quad \forall X = [x_{ij}] \in R(n,q),$$

where $a_X \in F_q^*$ depends on X; P is an invertible upper triangular matrix; π is an automorphism of the field F_q, $[\pi(x_{ij})]$ denotes the matrix whose (i,j)-entry is $\pi(x_{ij})$.

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1. Introduction

Let R be a commutative ring with identity 1. The zero-divisor graph of R, denoted by $\Gamma(R)$, is an undirected graph with vertices $Z^*(R)$, the set of nonzero zero-divisors of R, and for distinct elements $x, y \in Z^*(R)$, there is an edge joining x and y if and only if $xy = 0$. The concept of zero-divisor graph was first defined and studied for commutative rings by Beck in [7], and further studied by many authors (see, e.g., Akbari et al. [1]; Anderson et al. [2–5]; Axtell et al. [6]; Levy et al. [12]; Lucas [17]; Wu [24]). A lot of results about the diameter, the girth of $\Gamma(R)$ and so on have been obtained. The main idea of these researches is to study the interplay between the ring-theoretic properties of a commutative ring R and the graph-theoretic properties of $\Gamma(R)$. Redmond [20] extended the concept of zero-divisor graphs of commutative rings to zero-divisor graphs of noncommutative rings. Recently, zero-divisor graphs of matrix rings attracted a lot of attention. Akbari and Mohammadian [1] studied the problem of determining when the zero-divisor graphs of rings are isomorphic, given that the zero-divisor graphs of their matrix rings are isomorphic. I. Božić and Z. Petrović [8] further investigated the properties of (directed) zero-divisor graphs of matrix rings, studied the relation between the diameter of the zero-divisor graph of a commutative ring R and that of the matrix ring $\mathrm{Mat}_n(R)$. Li [13] proved that when studying $\Gamma(R)$, one can replace R with its total quotient ring, and they determined the girth of $\Gamma(\mathcal{T})$ of $\mathcal{T}$, triangular matrices over a commutative ring. The diameter, the girth and the bounds for the number of edges of $\Gamma(\mathcal{T})$ were given in [14].

Generally, determining the full automorphisms of a graph is an important however a difficult problem both in graph theory and in algebraic theory. Searching the literature, we find that little is known for the automorphisms of zero-divisor graphs of rings. In [3], Anderson and Livingston have shown that $\mathrm{Aut}(\Gamma(Z_n))$ is a direct product of symmetric groups for $n \geq 4$ a nonprime integer. For the noncommutative case, it was shown in [11] that $\mathrm{Aut}(\Gamma(R))$ is isomorphic to the symmetric group of degree $p+1$, when $R = \mathrm{Mat}_2(Z_p)$ (p is a prime). S. Park and J. Han [19] proved that $\mathrm{Aut}(\Gamma(R)) \cong S_{q+1}$ for $R = \mathrm{Mat}_2(F_q)$ with F_q an arbitrary finite field. It is slightly regrettable that the authors of [11] and [19] only concerned with the special case when $n = 2$. Till now nothing is known about the automorphisms of the zero-divisor graphs of $n \times n$ matrices with $n \geq 3$. This observation motivates us to do some work on this topic. It seems much difficult to determine the full automorphisms of $\Gamma(\mathrm{Mat}_n(q))$, so we focus our attention on the subgraph of $\Gamma(\mathrm{Mat}_n(q))$ induced by all rank one upper triangular matrices. Let $T(n, q)$ be the set of all $n \times n$ upper triangular matrices over F_q, $R(n, q)$ the set of all rank one matrices in $T(n, q)$. The subgraph of $\Gamma(\mathrm{Mat}_n(q))$ induced by all rank one upper triangular matrices is denoted by $\Gamma(R(n, q))$. The main theorem of this article proves that any graph automorphism of $\Gamma(R(n, q))$ with $n \geq 3$ can be decomposed into the product of an inner automorphism, a field automorphism and a local scalar multiplication.

For proving the main theorem of this paper we apply a technique from some recent papers which focused on determining the automorphisms of some graphs based on row