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The traces associated with a sharp tridiagonal system

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ABSTRACT

Let $\mathbb{K}$ denote a field and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let A, A^* denote a tridiagonal pair on V. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denote a standard ordering of the eigenvalues of A (resp. A^*) and for $0 \le i \le d$ let V_i (resp. V_i^*) be the eigenspace of A (resp. A^*) associated with θ_i (resp. θ_i^*). It is known that V_i, V_i^* have the same dimension. The tridiagonal pair A, A^* is said to be sharp whenever $\dim(V_0) = 1$. For $0 \le i \le d$, let E_i (resp. E_i^*) denote the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*). Then $\Phi = (A; E_0, E_1, \cdots, E_d; A^*; E_0^*, E_1^*, \cdots, E_d^*)$ is a tridiagonal system on V. We say Φ is sharp whenever the tridiagonal pair A, A^* is sharp. Assume Φ is sharp and let $\{\zeta_i\}_{i=0}^d$ denote the split sequence of Φ. The sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ is called the parameter array of Φ. Recently in [3] K. Nomura and P. Terwilliger proposed the following problem: Let Φ denote a sharp tridiagonal system. For $0 \le i \le d$ find each of

$$\operatorname{tr}(E_i E_0^*), \operatorname{tr}(E_i E_d^*), \operatorname{tr}(E_i^* E_0), \operatorname{tr}(E_i^* E_d)$$

in terms of the parameter array of Φ. In the present paper we solve this problem.

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1. Introduction

Throughout this paper $\mathbb{K}$ denotes a field and V denotes a vector space over $\mathbb{K}$ with finite positive dimension.

We begin by recalling the problem solved in this paper. We will use the following notation and terminology. These notation and terminology are standard. The reader is referred to the next section or [1,3] for the definitions.

Let A, A^* denote a tridiagonal pair on V with diameter d. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denote a standard ordering of the eigenvalues of A (resp. A^*). For $0 \le i \le d$ let V_i (resp. V_i^*) be the eigenspace of A (resp. A^*) associated with θ_i (resp. θ_i^*) and let E_i (resp. E_i^*) denote the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*). Then $\Phi = (A; E_0, E_1, \cdots, E_d; A^*; E_0^*, E_1^*, \cdots, E_d^*)$ is a tridiagonal system on V. Assume Φ is sharp and let $\{\zeta_i\}_{i=0}^d$ denote the split sequence of Φ. Let the sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ be the parameter array of Φ. Recently in [3] K. Nomura and P. Terwilliger proposed the following problem.

Problem 1.1. (See [3, Problem 12.8].) Assume Φ is a sharp tridiagonal system. Let the sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ be the corresponding parameter array of Φ. For $0 \le i \le d$ find each of

$$\operatorname{tr}(E_i E_0^*), \operatorname{tr}(E_i E_d^*), \operatorname{tr}(E_i^* E_0), \operatorname{tr}(E_i^* E_d)$$

in terms of the parameter array of Φ.

In this paper, for a given sharp tridiagonal system Φ, we calculate the traces of $E_i E_0^*$, $E_i E_d^*$, $E_i^* E_0$ and $E_i^* E_d$ for $0 \le i \le d$ in terms of the parameter array of Φ. Thus Problem 1.1 is solved.

This paper is organized as follows. In Section 2, we recall some definitions and basic terminologies for tridiagonal pairs and tridiagonal systems. Also we recall some notation which will be used in the expressions of the theorem. In Section 3, we give the main theorem and its proof.

2. Preliminaries

We will use the following terms. Let $\operatorname{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all $\mathbb{K}$-linear transformations from V to V. For $A \in \operatorname{End}(V)$ and for a subspace $W \subseteq V$, we call W an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case we call θ the *eigenvalue* of A associated with W. We say A is *diagonalizable* whenever V is spanned by the eigenspaces of A. We now recall the notion of a tridiagonal pair.

Definition 2.1. (See [1, Definition 1.1].) By a *tridiagonal pair* on V, we mean an ordered pair of elements A, A^* taken from $\operatorname{End}(V)$ that satisfy (i)–(iv) below.