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Regular sparse anti-magic squares with the second maximum density ☆

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ABSTRACT

Sparse anti-magic squares are useful in constructing vertex-magic labelings for bipartite graphs. An $n \times n$ array based on $\{0, 1, \cdots, nd\}$ is called *a sparse anti-magic square of order n with density d* ($d < n$), denoted by SAMS(n, d), if its row-sums, column-sums and two main diagonal sums form a set of $2n+2$ consecutive integers. An SAMS(n, d) is called *regular* if there are d positive entries in each row, each column and each main diagonal. In this paper, we investigate the existence of regular sparse anti-magic squares with the second maximum density, i.e., $d = n - 2$, and it is shown that there exists a regular SAMS$(n, n-2)$ if and only if $n \geq 4$.

1. Introduction

Magic squares and their various generalizations have been objects of interest for many centuries and in many cultures. A lot of work had been done on the constructions of

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magic squares, for more details, the interested reader may refer to [1–4] and the references therein.

An anti-magic square of order n is an $n \times n$ array with entries consisting of n^2 consecutive nonnegative integers such that the row-sums, column-sums and two main diagonal sums constitute a set of consecutive integers. Usually, the main diagonal from upper left to lower right is called *the left diagonal*, another is called *the right diagonal*. The existence of an anti-magic square has been solved completely by Cormie et al. (see [5,6]). It was shown that there exists an anti-magic square of order n if and only if $n \geq 4$.

For positive integers n, d ($d < n$), an $n \times n$ array based on $\{0, 1, \cdots, nd\}$ is called *sparse magic square of order* n *with density* d, denoted by $\mathrm{SMS}(n, d)$, if its row-sums, column-sums and two main diagonal sums are the same. An $\mathrm{SMS}(n, d)$ is called *regular* if there exist d elements in each row, each column and each main diagonal. The existence of a regular $\mathrm{SMS}(n, d)$ has been solved completely by Li et al. [7]. It was shown that there exists a regular $\mathrm{SMS}(n, d)$ if and only if $d \geq 3$ when n is odd and $d \geq 4$ when n is even.

Sparse anti-magic squares are generalization of anti-magic squares. For positive integers n, d ($d < n$), let A be an $n \times n$ array with entries consisting of $0, 1, \cdots, nd$ and let S_A be the set of row-sums, column-sums and two main diagonal sums of A. We call S_A the *sum set* of A. Then A is called a *sparse anti-magic square of order* n *with density* d, denoted by $\mathrm{SAMS}(n, d)$, if S_A consists of $2n + 2$ consecutive integers. In [8], an $\mathrm{SAMS}(n, d)$ is also called a *sparse totally anti-magic square*. An $\mathrm{SAMS}(n, d)$ is called *regular* if all of its rows, columns and two main diagonals contain d positive entries.

Sparse anti-magic squares are useful in graph theory. For example, they can be used to construct a vertex-magic total labeling for bipartite graphs, see [8] and the references therein.

Quite recently, the authors Chen et al. [9] proved the following result.

Theorem 1.1. *There exists a regular SAMS*$(n, n-1)$ *if and only if* $n \geq 4$.

In this paper, we investigate the existence of a regular sparse anti-magic square with $d = n - 2$. It is not difficult to see that there is no $\mathrm{SAMS}(3, 1)$. So to consider the existence of a regular $\mathrm{SAMS}(n, n-2)$, we need only to consider the case of $n \geq 4$. We shall prove the following.

Theorem 1.2. *There exists a regular SAMS*$(n, n-2)$ *if and only if* $n \geq 4$.

2. Regular SAMS$(n, n-2)$ with n even

In this section, we shall provide construction of sparse anti-magic squares based on quasi sparse anti-magic squares and prove that there exists a regular $\mathrm{SAMS}(n, n-2)$ for any $n \geq 4$ and n is even.