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The skeleton of acyclic Birkhoff polytopes

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For a fixed tree T with n vertices the corresponding acyclic Birkhoff polytope $\Omega_n(T)$ consists of doubly stochastic matrices having support in positions specified by T. This is a face of the Birkhoff polytope Ω_n (which consists of all $n \times n$ doubly stochastic matrices). The skeleton of $\Omega_n(T)$ is the graph where vertices and edges correspond to those of $\Omega_n(T)$, and we investigate some properties of this graph. In particular, we characterize adjacency of pairs of vertices, compute the minimum degree of a vertex and show some properties of the maximum degree of a vertex in the skeleton. We also determine the maximum degree for certain classes of trees, including paths, stars and caterpillars.

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1. Introduction

Let $T := T(V(T), E(T))$ be a tree with the vertex set $V := V(T) = \{v_1, v_2, \dots, v_n\}$ and the set of edges $E := E(T) = \{e_1, e_2, \dots, e_{n-1}\}$. For each k, $1 \le k \le n-1$, the edge $e_k = \{v_i, v_j\}$ is simply denoted by $v_i v_j$ or by the indices of its end vertices, that is, $e_k = ij$ and in this case, we say that v_i is adjacent to v_j. The *neighbors* of v_k are its adjacent vertices and the set of neighbors of v_k, called the *neighborhood* of v_k, is denoted by $N_T(v_k)$. The degree of a vertex v_k is $d(v_k) = d_{v_k} = |N_T(v_k)|$. Moreover, $E(v_k)$ is the set of edges in T incident to v_k. A *pendant edge* has an end vertex with degree 1 which is known as a *pendant vertex* or a *leaf* of the tree. A vertex in T which is not a leaf is called an *inner vertex*. The *path* with n vertices, P_n, is a tree in which every vertex has degree 2 except the terminal vertices. The *length* of P_n, denoted by $|E(P_n)|$, is its number of edges. For more basic definitions and notations of graphs (particularly of trees), see [3].

Consider a Euclidean space S with inner product $\langle\cdot,\cdot\rangle$, and let C be a convex set in S. A subset $F \subseteq C$ is a *face* of C if F is a convex set and satisfies the following property: whenever $x, y \in C$ and $(1-\lambda)x + \lambda y \in F$ for some $0 < \lambda < 1$, then $x, y \in F$. A *halfspace* is a set $H^- = \{x \in S : \langle a, x\rangle \le b\}$ where $a \in S$ is nonzero and $b \in \mathbb{R}$. Then $H = \{x \in S : \langle a, x\rangle = b\}$ is the corresponding *hyperplane* (an affine set of dimension $\dim S - 1$). A *polyhedron* in S is the intersection of a finite number of halfspaces. If $C \subseteq H^-$ and $H \cap C$ is nonempty, we call H a *supporting hyperplane* of C. An *exposed face* of C is the intersection of C and one of its supporting hyperplanes. In general every exposed face is a face, but the converse may not hold. For polyhedra, however, faces and exposed faces coincide. Note that the support of a matrix A is the set of the positions of the nonzero entries of A. For more details concerning convex analysis (especially polyhedral theory), see [14,17].

A real $n \times n$ matrix is *doubly stochastic* if it is a nonnegative matrix and each row and column sum is 1 (see [4,7–9]). The set of all doubly stochastic matrices of order n is denoted by Ω_n, and a classical result due to Birkhoff and von Neumann [2,16] says that Ω_n is a polytope whose extreme points are the permutation matrices (see also [7]). There is a correspondence between doubly stochastic and doubly sub-stochastic matrices (i.e., nonnegative matrices where each row and column sum is at most 1) and matchings in bipartite graphs and related polytopes. The polytope Ω_n is investigated in detail in [4–6], and different subpolytopes are discussed in e.g. [8–10]. The recent paper [1] treats general properties of fractional perfect matching polytopes in detail, including the facial structure. The present paper, however, can be viewed as a study of matching (not perfect) polytopes associated with trees, and with focus on minimum and maximum degrees.

Let T be a tree with n vertices. The acyclic Birkhoff polytope, $\Omega_n(T)$, introduced in [8], is the set of doubly stochastic matrices A such that each positive entry of A is either on the diagonal or in a position that corresponds to an edge of T, i.e., in the positions (i, j) and (j, i), where ij is an edge of T. The diagonal entries of A correspond to the vertices of T. It was shown in [8] that each matrix $A \in \Omega_n(T)$ is symmetric.