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Modular representations of Heisenberg algebras

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ABSTRACT

Let F be an arbitrary field and let $\mathfrak{h}(n)$ be the Heisenberg algebra of dimension $2n+1$ over F. It was shown by Burde that if F has characteristic 0 then the minimum dimension of a faithful $\mathfrak{h}(n)$-module is $n+2$. We show here that his result remains valid in prime characteristic p, as long as $(p,n) \neq (2,1)$.

We construct, as well, various families of faithful irreducible $\mathfrak{h}(n)$-modules if F has prime characteristic, and classify these under suitable assumptions on F. Applications to matrix theory are given.

1. Introduction

Let F be an arbitrary field. For $n \geq 1$, let $\mathfrak{h}(n,F)$, or just $\mathfrak{h}(n)$, stand for the Heisenberg algebra of dimension $2n+1$ over F. This is a 2-step nilpotent Lie algebra with 1-dimensional center. It was shown by Burde [1] that when F has characteristic 0 the minimum dimension of a faithful $\mathfrak{h}(n)$-module is $n+2$. Further results on low dimensional imbeddings of nilpotent Lie algebras when $\mathrm{char}(F)=0$ can be found in [1,2,4].

Here we extend Burde's result to arbitrary fields by showing that the minimum dimension of a faithful $\mathfrak{h}(n)$-module is always $n+2$, except only when $n=1$ and $\mathrm{char}(F)=2$.

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See Section 3 for details. Our main tool if F has prime characteristic is the classification of faithful irreducible $\mathfrak{h}(n)$-modules under suitable assumptions on F, as found in Theorem 2.2. This classification is known when F is algebraically closed (see [6], page 149). For an arbitrary field of prime characteristic there exist, in general, families of faithful irreducible $\mathfrak{h}(n)$-modules that fall outside of this classification, some of which are studied in Sections 4 and 5. Applications to matrix theory arising from the representation theory of $\mathfrak{h}(n)$ can be found in Section 2.

We fix throughout a symplectic basis $x_1, \dots, x_n, y_1, \dots, y_n, z$ of $\mathfrak{h}(n)$, i.e., one with multiplication table $[x_i, y_i] = z$. Clearly, a representation $R : \mathfrak{h}(n) \to \mathfrak{gl}(V)$ is faithful if and only if $R(z) \neq 0$. All representations will be finite dimensional, unless otherwise mentioned. If $R : \mathfrak{g} \to \mathfrak{gl}(V)$ and $T : \mathfrak{g} \to \mathfrak{gl}(V)$ are representations of a Lie algebra $\mathfrak{g}$, we refer to T and R as equivalent if there is $\Omega \in \mathrm{Aut}(\mathfrak{g})$ such that T is similar to $R \circ \Omega$.

2. Faithful irreducible representations of $\mathfrak{h}(n)$

Proposition 2.1. *Let $F[X_1, \dots, X_n]$ be the polynomial algebra in n commuting variables $X_1, \dots, X_n$ over F. For $q \in F[X_1, \dots, X_n]$, let m_q be the linear endomorphism "multiplication by q" of $F[X_1, \dots, X_n]$. Let $\alpha, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n \in F$, where $\alpha \neq 0$. Then*

(1) $F[X_1, \dots, X_n]$ is a faithful $\mathfrak{h}(n)$-module via

$$z \mapsto \alpha \cdot I, \qquad x_i \mapsto \beta_i \cdot I + \alpha \cdot \partial/\partial X_i, \qquad y_i \mapsto \gamma_i \cdot I + m_{X_i}.$$

(2) $F[X_1, \dots, X_n]$ is irreducible if and only if F has characteristic 0.

(3) Suppose F has prime characteristic p. Then $(X_1^p, \dots, X_n^p)$ is an $\mathfrak{h}(n)$-invariant subspace of $F[X_1, \dots, X_n]$ and

$$V_{\alpha,\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n} = F[X_1, \dots, X_n]/\big(X_1^p, \dots, X_n^p\big)$$

is a faithful irreducible $\mathfrak{h}(n)$-module of dimension p^n. Moreover, $V_{\alpha,\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n}$ is isomorphic to $V_{\alpha',\beta'_1,\dots,\beta'_n,\gamma'_1,\dots,\gamma'_n}$ if and only if $\alpha = \alpha'$ and $\beta_i = \beta'_i, \gamma_i = \gamma'_i$ for all $1 \leq i \leq n$. Furthermore, $V_{\alpha,\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n}$ is equivalent to $V_{1,0,\dots,0}$.

Proof. This is straightforward. □

Theorem 2.2. *Suppose F has prime characteristic p. Let $R : \mathfrak{h}(n) \to \mathfrak{gl}(V)$ be a faithful irreducible representation. Assume each $z, x_1, \dots, x_n, y_1, \dots, y_n$ acts on V with at least one eigenvalue in F, say $\alpha, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n \in F$, respectively (this is automatic if F is algebraically closed). Then V is isomorphic to $V_{\alpha,\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n}$.*

Proof. We divide the proof into various steps.

Step 1. *$R(z) = \alpha \cdot I$, where $\alpha \neq 0$.*