Linear Algebra and its Applications 457 (2014) 76–92

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Some unitary similarity invariant sets preservers of skew Lie products ☆

Jianlian Cui [a,*], Qiting Li [a], Jinchuan Hou [b,c], Xiaofei Qi [c]

[a] Department of Mathematical Sciences, Tsinghua University, Beijing 100084, PR China
[b] Department of Mathematics, Taiyuan University of Technology, Taiyuan 030024, PR China
[c] Department of Mathematics, Shanxi University, Taiyuan 030006, PR China

ARTICLE INFO

Article history:
Received 20 November 2013
Accepted 6 May 2014
Available online 29 May 2014
Submitted by P. Semrl

MSC:
47B48
46L10

Keywords:
Numerical range
Numerical radius
Pseudo-spectrum
Skew Lie products

ABSTRACT

Let H and K be complex separable Hilbert spaces with dimensions at least three, and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. Let $\Delta(\cdot)$ denote $W(\cdot)$ or $\sigma_\varepsilon(\cdot)$, where, for A, $W(A)$ stands for the numerical range of $A \in \mathcal{B}(H)$ and $\sigma_\varepsilon(A)$ the ε-pseudospectrum of A. It is shown that a bijective map (no algebraic structure assumed) $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ satisfies that $\Delta(AB - BA^*) = \Delta(\Phi(A)\Phi(B) - \Phi(B)\Phi(A)^*)$ for all $A, B \in \mathcal{B}(H)$ if and only if there exists a unitary operator $U \in \mathcal{B}(H, K)$ such that $\Phi(A) = \mu UAU^*$ for all $A \in \mathcal{B}(H)$, where $\mu \in \{-1, 1\}$. If $\Delta(\cdot) = W(\cdot)$, then the injectivity assumption on Φ can be omitted.

☆ This project was supported by National Natural Science Foundation of China (No. 11271217, No. 11171249, No. 11101250).

* Corresponding author.
E-mail addresses: jcui@math.tsinghua.edu.cn (J. Cui), jinchuanhou@aliyun.com (J. Hou), qixf1980@126.com (X. Qi).

http://dx.doi.org/10.1016/j.laa.2014.05.009
0024-3795/

1. Introduction

Let $\mathcal{A}$ be a $*$-ring. For any $A, B \in \mathcal{A}$, the product $AB - BA^*$ is called the skew Lie product of A and B. This product is playing a more and more important role in some research topics, and its study has recently attracted many authors' attention (see, for example, [2–6,12,14–16]). Denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a real or complex Hilbert space H of dimension greater than 1. Motivated by the theory of rings (and algebras) equipped with the Lie product $[A, B] = AB - BA$ or the Jordan product $A \circ B = AB + BA$, Molnár in [12] initiated a systematic study of the skew Lie product, and showed that if $\mathcal{N} \subseteq \mathcal{B}(H)$ is an ideal, then $\mathcal{N} = \operatorname{span}\{AB - BA^* \mid A \in \mathcal{N}, B \in \mathcal{B}(H)\} = \operatorname{span}\{AB - BA^* \mid A \in \mathcal{B}(H), B \in \mathcal{N}\}$; in particular, every element of $\mathcal{B}(H)$ is a finite sum of $TS - ST^*$ type operators. Later, Brešar and Fošner [2] generalized the above results in [12] to rings with involution in different directions.

Let $\mathcal{A}$ and $\mathcal{B}$ be two von Neumann algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ be a map. If $\Phi(AB - BA^*) = \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^*$ for all $A, B \in \mathcal{A}$, we say that Φ preserves skew Lie products; if $\Phi(A)\Phi(B) - \Phi(B)\Phi(A)^* = AB - BA^*$ for all $A, B \in \mathcal{A}$, we say that Φ preserves strong skew Lie products. For $A, B \in \mathcal{A}$, if $AB = BA^*$ implies that $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)^*$, we say that Φ preserves zero skew Lie products. Clearly, every skew Lie product preserving map is zero skew Lie product preserving. Cui and Li [5] proved that, if $\mathcal{A}$ and $\mathcal{B}$ are factor Von Neumann algebras and Φ is a bijective map preserving skew Lie products, then Φ is a $*$-ring isomorphism. Cui and Park [6] proved that, if $\mathcal{A}$ and $\mathcal{B}$ are factor Von Neumann algebras and Φ is a surjective map preserving strong skew Lie products, then Φ is of the form $\Phi(A) = \Psi(A) + h(A)I$ for all $A \in \mathcal{A}$, where $\Psi : \mathcal{A} \to \mathcal{B}$ is a linear bijective map preserving strong skew Lie products and $h : \mathcal{A} \to \mathbb{R}$ is a real functional satisfying $h(0) = 0$; in particular, when $\mathcal{A} = \mathcal{B} = \mathcal{B}(H)$, the above map Ψ is the identity map on $\mathcal{B}(H)$. Qi and Hou [14] generalized and improved the result in [6] to von Neumann algebras without central summands of type I_1; they characterized also the corresponding maps on prime involution rings and prime involution algebras. When $\mathcal{A} = \mathcal{B}(H)$ and $\mathcal{B} = \mathcal{B}(K)$ with H and K being two complex Hilbert spaces, as a special case of results in [4], Cui and Hou characterized the linear bijective maps $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ preserving zero skew Lie products, and proved that such a Φ is of the form $\Phi(A) = cUAU^*$ for all $A \in \mathcal{B}(H)$, where $U \in \mathcal{B}(H, K)$ is unitary and c is a nonzero real number. For the maps on von Neumann algebras, even if under the assumption that the involved maps are additive, it seems considerably difficult to characterize maps preserving zero skew Lie products.

The purpose of this paper is to discuss the maps that preserve some properties of skew Lie products on factor von Neumann algebras. More exactly, we will respectively characterize the maps preserving numerical range and the maps preserving pseudo-spectrum of skew Lie products of operators on $\mathcal{B}(H)$ (see Theorem 2.2 and Theorem 3.3), and characterize the additive maps preserving numerical radius of skew Lie products on factor von Neumann algebras (see Theorem 4.1). Our results reveal that, for a map