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# Local spectrum linear preservers at non-fixed vectors

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A B S T R A C T

For a complex $n \times n$ matrix $T$ and a vector $x \in \mathbb{C}^n$, we denote by $\sigma_T(x)$ (respectively, by $r_T(x)$) the local spectrum (respectively, the local spectral radius) of $T$ at $x$. We prove that $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ linear has the property that for each $T \in \mathcal{M}_n$ there exists a nonzero $x_T \in \mathbb{C}^n$ such that $\sigma_{\varphi(T)}(x_T) = \sigma_T(x_T)$ if, and only if, there exists $A \in \mathcal{M}_n$ invertible such that either $\varphi(T) = ATA^{-1}$ for each $T \in \mathcal{M}_n$, or $\varphi(T) = AT^tA^{-1}$ for each $T \in \mathcal{M}_n$. Modulo a multiplication by a unimodular complex number, we arrive at the same conclusion by supposing that for each $T \in \mathcal{M}_n$ there exists a nonzero $x_T \in \mathbb{C}^n$ such that $r_{\varphi(T)}(x_T) = r_T(x_T)$.



## 1. Introduction and statement of the main results

For a fixed integer $n \geq 1$, let us denote by $\mathcal{M}_n$ the space of all $n \times n$ matrices over the complex field $\mathbb{C}$. For $T \in \mathcal{M}_n$, by $\sigma(T)$ we shall denote its spectrum, that is the set of all eigenvalues of $T$ without counting multiplicities, and by $\rho(T)$ its spectral radius, that is the maximum modulus of $\sigma(T)$. By a result of Marcus and Moyls [12, Theorem 3] and a density argument, if $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ is linear and $\sigma(\varphi(T)) = \sigma(T)$ for each $T$,

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then $\varphi$ is either an automorphism or an anti-automorphism of $\mathcal{M}_n$: there exists $A \in \mathcal{M}_n$ invertible such that either

$$\varphi(T) = ATA^{-1} \quad (T \in \mathcal{M}_n) \quad \text{or} \quad \varphi(T) = AT^tA^{-1} \quad (T \in \mathcal{M}_n). \tag{1}$$

(Throughout this paper, by $T^t$ we shall denote the transpose of the matrix $T$.)

Akbari and Aryapoor proved that we arrive at the same conclusion by supposing only that $\sigma(\varphi(T)) \cap \sigma(T) \neq \emptyset$ for each $T$ [1, Corollary 3]. This can be used in order to obtain a new proof for the fact (see [3, Proposition 2]) that a unital linear map $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ satisfying $\rho(\varphi(T)) = \rho(T)$ for each $T$ is also of the form (1). Any such map $\varphi$ is bijective [13] and therefore preserves the peripheral spectrum [8]. Thus for each $T$ the points $\lambda \in \sigma(\varphi(T))$ for which $|\lambda| = \rho(T)$ are exactly the points $\lambda \in \sigma(T)$ with $|\lambda| = \rho(T)$. Since such a $\lambda$ exists always, we have that $\sigma(\varphi(T))$ and $\sigma(T)$ have always at least one common element, allowing us to use [1, Corollary 3].

The above results have been generalized in different directions over the last years, and one way to do it is to replace the spectrum function/spectral radius by the local spectrum/local spectral radius at vectors from $\mathbb{C}^n$. (We refer the reader to the books [11] and [15] for the standard notions from local spectral theory.) For $T \in \mathcal{M}_n$ and $x \in \mathbb{C}^n$, we denote by $\sigma_T(x)$ (respectively $r_T(x)$) the local spectrum (respectively, the local spectral radius) of $T$ at $x$. Fixing $x_0 \in \mathbb{C}^n \backslash \{0\}$, González and Mbekhta proved in [9] that if $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ is linear and $\sigma_{\varphi(T)}(x_0) = \sigma_T(x_0)$ for every $T \in \mathcal{M}_n$, there exists then an invertible matrix $A$ such that $A(x_0) = x_0$ and $\varphi(T) = ATA^{-1}$ for each $T \in \mathcal{M}_n$. The aim of this paper is to study the same type of linear preserver problem in the case when the vector $x_0$ is not fixed, but varies with $T$. A characterization of such maps is given by our first result.

**Theorem 1.** *Let $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ be a linear map. Then the following statements are equivalent:*

*(i) For each $T \in \mathcal{M}_n$ there exists a nonzero vector $x_T \in \mathbb{C}^n$ such that*

$$\sigma_{\varphi(T)}(x_T) = \sigma_T(x_T). \tag{2}$$

*(ii) For each $T \in \mathcal{M}_n$ there exists a nonzero vector $x_T \in \mathbb{C}^n$ such that*

$$\sigma_{\varphi(T)}(x_T) \cap \sigma_T(x_T) \neq \emptyset. \tag{3}$$

*(iii) The map $\varphi$ is of the form (1).*

The case of the local spectral radius preservers at some fixed vector $x_0 \in \mathbb{C}^n \backslash \{0\}$ was characterized by Bourhim and Miller in [5]. They proved that if $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ is linear and $r_{\varphi(T)}(x_0) = r_T(x_0)$ for every $T \in \mathcal{M}_n$, there exist then an invertible matrix $A$ and