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A singular M-matrix perturbed by a nonnegative rank one matrix has positive principal minors; is it D-stable?

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The positive stability and D-stability of singular M-matrices, perturbed by (non-trivial) nonnegative rank one perturbations, is investigated. In special cases positive stability or D-stability can be established. In full generality this is not the case, as illustrated by a counterexample. However, matrices of the mentioned form are shown to be P-matrices.

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1. Introduction

1.1. Problem formulation

Let $H \in \mathbb{C}^{n \times n}$. Let $\rho(H)$ denote the spectral radius of H, i.e.

$$\rho(H) = \max\{|\lambda| : \lambda \in \mathbb{C}, \det(\lambda I - H) = 0\}.$$

We consider the matrix $A := \rho(H)I - H$. Let $v, w \in \mathbb{C}^n$. In this paper we deal with the stability properties of the matrix $B = \rho(H)I - H + vw^\star = A + vw^\star$. In particular, we may ask the following questions:

Problem 1.1.

(i) Under what conditions is B strictly positive stable?
(ii) Let $C \in \mathbb{C}^{n \times n}$. Under what conditions is CB strictly positive stable?

Our primary interest is in the case where H is real nonnegative, in which case A is a singular M-matrix, and the perturbation vw^T is positive, i.e. a matrix having only positive entries.

We encountered this problem in the process of studying stability of a particular type of ordinary differential equation. This motivation is discussed briefly in Section 1.3, and with some more attention to detail in Appendix A. We will provide a few general observations in Section 1.4. These observations are not restricted to the class of nonnegative matrices, and will help us to eliminate some trivial cases. Still by means of introduction, in Section 1.5 we will pay attention to the elementary case where H is not assumed to be nonnegative but instead symmetric (or more generally, normal), and the rank one perturbation symmetric.

The nonnegative case is more challenging. Our main question is whether a matrix of the mentioned form is D-stable, i.e. stable even when multiplied by any positive diagonal matrix. This problem turns out to be difficult, and we cannot yet conclusively answer this question. However we are able to show stability or even D-stability in certain special cases, which may be considered as the main results of this paper (Section 2). A counterexample is provided to show that we cannot hope to establish D-stability for the general class of singular M-matrices with positive rank one perturbations. To provide direction for further research, we state as a conjecture that if A is a singular and symmetric M-matrix, with geometrically simple eigenvalue 0, and if vw^T is a positive rank one perturbation, then $A + vw^T$ is D-stable; see Section 2.1 for the counterexample and the conjecture. A result that is potentially useful in this direction, and interesting in its own right, is that all matrices in this class are P-matrices, i.e. matrices with positive principal minors (Section 2.2).

The problem is part of a wider context: in the last two decades the study of eigenvalues and Jordan structure of rank-one perturbations of matrices has seen rapid development.