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Variances and determinantal profiles of orientations ☆

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ABSTRACT

Given a simple graph G, let X be the random variable which is the determinant of the (oriented) adjacency matrix of an orientation of G. It is known that the expectation $\mathrm{E}(X)$ equals the number of perfect matchings of G. In this paper we give a graphical interpretation of the variance $\mathrm{Var}(X)$. We also give complete determinantal profiles of several classes of graphs, including wheels, fans, and general books.

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1. Introduction

1.1. Determinants of orientations

Let $G = (V, E)$ be a simple graph with $V = \{v_1, v_2, \dots, v_n\}$ and D be an orientation of G. The (oriented) *adjacency matrix* of D is a square matrix whose rows and columns are both indexed by the vertices $v_1, v_2, \dots, v_n$, and the (i, j)-entry is $1, -1$ or 0 if there is an edge oriented from v_i to v_j, from v_j to v_i, or no edge in between, respectively. For simplicity, $D = (d_{ij})$ is used to denote both an orientation and its adjacency matrix.

A *perfect matching*, or 1-*factor*, of G is a set of edges in which each vertex of G belongs to exactly one edge. Denote by $M(G)$ the number of 1-factors of G.

Suppose that each edge of G is independently oriented either way with probability $\frac{1}{2}$. For each orientation of G we have a square matrix D, hence a determinant $\det(D)$. Let $X := \det(D)$ be a random variable. There is a nice graphical interpretation of the expectation $\mathrm{E}(X)$.

Theorem 1.1. *(See [1], Exercise 10.10, page 466.) We have*

$$\mathrm{E}(X) = M(G).$$

In other words, the "average determinant" of all orientations equals the number of perfect matchings. Hence it is natural to find an interpretation of the variance $\mathrm{Var}(X)$ in terms of graph parameters. A 2-*factor* of a graph G is a spanning 2-regular subgraph of G. A *deformed* 2-*factor* of G is a spanning subgraph of G where each component is either a cycle or an edge. Note that a cycle consists of at least three vertices. We say that a deformed 2-factor is *special* if it contains at least one cycle and each cycle is of even length. Let $\mathcal{H}(G)$ denote the set of special deformed 2-factors of G.

Let $\mathcal{C}(G)$ be a maximum set of vertex-disjoint cycles of a graph G and let $c(G) := |\mathcal{C}(G)|$. Hence for $H \in \mathcal{H}(G)$, $c(H)$ is the number of cycles of H.

The main result of this paper is the following interpretation of $\mathrm{Var}(X)$.

Theorem 1.2. *We have*

$$\mathrm{Var}(X) = \sum_{H \in \mathcal{H}(G)} 6^{c(H)} - 2^{c(H)}.$$

A deformed 2-factor is also called a *linear subgraph* in the literature [3]. We use the name deformed 2-factor to emphasize that $\mathrm{E}(X)$ involves 1-factors while $\mathrm{Var}(X)$ involves a variation of 2-factors.

We illustrate the theorem by an example to show that for nice graphs our result can reduce the computation of $\mathrm{Var}(X)$ significantly.

Example. Let P_n denote the path of n vertices. Consider the graph $G = P_2 \square P_4$, the Cartesian product of P_2 and P_4. Exhausting computer check shows that among all