

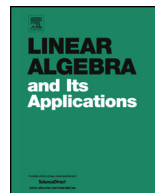


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## Some ways of constructing Furuta-type inequalities



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## ABSTRACT

Let  $\sigma_h$  be an operator mean associated with an operator monotone function  $h$  and let  $A, B$  be positive operators. We investigate the following Furuta-type inequality: For some fixed continuous function  $h$ ,

$$A \geq B > 0 \Rightarrow A^{-r} \sigma_{\varphi_r} X_h^{-r} \leq B \quad (r \geq 1),$$

where  $X_h$  is the positive solution of  $h(X) = B^{-1}$ . The map  $(h, r) \mapsto \varphi_r$  plays a central role in constructing a Furuta-type inequality, similar to the role of the map  $(p, r) \mapsto \frac{r+1}{p+r}$  in the following part of the Furuta inequality:  $A \geq B > 0, p, r \geq 1 \Rightarrow (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq (B^{p+r})^{\frac{1+r}{p+r}}$ . We obtain this map  $(h, r) \mapsto \varphi_r$  from the following Ando–Hiai type inequality:

$$A, B > 0, \quad A \sigma_h B \leq 1 \Rightarrow A^r \sigma_h B^r \leq 1 \quad (r \geq 1).$$

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## 1. Introduction

Let  $\mathcal{H}$  be a complex Hilbert space with inner product  $\langle \cdot | \cdot \rangle$ . Here, a bounded linear operator  $A$  is said to be positive semidefinite (denoted by  $A \geq 0$ ) if  $\langle Ax | x \rangle \geq 0$  for all

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$x \in \mathcal{H}$  and  $A \geq B$  means that  $A - B$  is positive semidefinite. Moreover,  $A$  is said to be positive definite (denoted by  $A > 0$ ) if  $A$  is positive semidefinite and invertible [3].

In [6], Furuta proved the following inequality: If  $A \geq B \geq 0$ , then for each  $r \geq 0$ ,

$$A^{\frac{p+r}{q}} \geq \left( A^{\frac{r}{2}} B^p A^{\frac{r}{2}} \right)^{\frac{1}{q}} \tag{1.1}$$

and

$$\left( B^{\frac{r}{2}} A^p B^{\frac{r}{2}} \right)^{\frac{1}{q}} \geq B^{\frac{p+r}{q}} \tag{1.2}$$

hold for  $p$  and  $q$  such that  $p \geq 0$  and  $q \geq 1$  with  $(1+r)q \geq p+r$ . This inequality is called the Furuta inequality and has been widely investigated in the literature. Tanahashi [11] showed that for fixed  $p, r$ , the best lower bound of  $q \geq 1$  satisfying (1.1) or (1.2) is  $\frac{p+r}{1+r}$ . Thus it follows from the Löwner–Heinz inequality [7,8] that the Furuta inequality is essential in the case when  $p, r \geq 1$  and  $q = \frac{p+r}{1+r}$ . So we can assume that when we discuss the Furuta inequality, we mean this mapping:  $(p, r) \mapsto \frac{1+r}{p+r}$ . In [12], Uchiyama used his product theorem to establish some operator inequalities which contain the Furuta inequality.

A continuous function  $f$  from  $[0, \infty)$  into itself is said to be operator monotone on  $[0, \infty)$  if for two positive semidefinite operators  $A$  and  $B$ , the inequality  $A \geq B$  implies  $f(A) \geq f(B)$ . An operator monotone function  $f$  on  $[0, \infty)$  is said to be normal if  $f(1) = 1$ . Throughout this paper,  $OM_+^1$  stands for the set of normalized operator monotone functions from  $[0, \infty)$  into itself. Let  $A, B$  be positive invertible operators on  $\mathcal{H}$ . An operator mean  $A\sigma B$  associated with  $f \in OM_+^1$  is defined as

$$A\sigma B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

and its properties were studied in [4]. In particular, the weighted geometric mean  $\#_\alpha$  associated with  $f(t) = t^\alpha$  ( $0 \leq \alpha \leq 1$ ) has often been studied separately from the point of view of functional analysis, statistics, quantum information, or other fields. Ando and Hiai [2] proved the following theorem concerning weighted geometric means: If  $A, B$  are positive invertible operators with  $A\#_\alpha B \leq 1$ , then  $A^r\#_\alpha B^r \leq 1$  for all  $r \geq 1$ . This theorem is referred to as the Ando–Hiai inequality. In [5], Fujii and Kamei showed that the Ando–Hiai inequality implies the following essential part of the Furuta inequality:

$$A \geq B > 0 \quad \Rightarrow \quad B^p \#_{\frac{p-1}{p+r}} A^{-r} (= A^{-r} \#_{\frac{1+r}{p+r}} B^p) \leq B \quad (r, p \geq 1). \tag{1.3}$$

In the present paper, we study a way to construct the following Furuta-type inequality:

$$A \geq B > 0 \quad \Rightarrow \quad X^{-r} \sigma_{\varphi_r} A^{-r} \leq B \quad (r \geq 1), \tag{1.4}$$

where  $X$  is the positive solution of  $h(X) = B^{-1}$  for some function  $h$ . From the following Ando–Hiai type inequality:

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