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Some ways of constructing Furuta-type inequalities

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Let σ_h be an operator mean associated with an operator monotone function h and let A, B be positive operators. We investigate the following Furuta-type inequality: For some fixed continuous function h,

$$A \geq B > 0 \quad \Rightarrow \quad A^{-r} \sigma_{\varphi_r} X_h^{-r} \leq B \quad (r \geq 1),$$

where X_h is the positive solution of $h(X) = B^{-1}$. The map $(h, r) \mapsto \varphi_r$ plays a central role in constructing a Furuta-type inequality, similar to the role of the map $(p, r) \mapsto \frac{r+1}{p+r}$ in the following part of the Furuta inequality: $A \geq B > 0, p, r \geq 1 \Rightarrow (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq (B^{p+r})^{\frac{1+r}{p+r}}$. We obtain this map $(h, r) \mapsto \varphi_r$ from the following Ando–Hiai type inequality:

$$A, B > 0, \quad A \sigma_h B \leq 1 \quad \Rightarrow \quad A^r \sigma_h B^r \leq 1 \quad (r \geq 1).$$

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$. Here, a bounded linear operator A is said to be positive semidefinite (denoted by $A \geq 0$) if $\langle Ax \mid x \rangle \geq 0$ for all

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$x \in \mathcal{H}$ and $A \geq B$ means that $A - B$ is positive semidefinite. Moreover, A is said to be positive definite (denoted by $A > 0$) if A is positive semidefinite and invertible [3].

In [6], Furuta proved the following inequality: If $A \geq B \geq 0$, then for each $r \geq 0$,

$$A^{\frac{p+r}{q}} \geq \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \tag{1.1}$$

and

$$\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq B^{\frac{p+r}{q}} \tag{1.2}$$

hold for p and q such that $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$. This inequality is called the Furuta inequality and has been widely investigated in the literature. Tanahashi [11] showed that for fixed p, r, the best lower bound of $q \geq 1$ satisfying (1.1) or (1.2) is $\frac{p+r}{1+r}$. Thus it follows from the Löwner–Heinz inequality [7,8] that the Furuta inequality is essential in the case when $p, r \geq 1$ and $q = \frac{p+r}{1+r}$. So we can assume that when we discuss the Furuta inequality, we mean this mapping: $(p, r) \mapsto \frac{1+r}{p+r}$. In [12], Uchiyama used his product theorem to establish some operator inequalities which contain the Furuta inequality.

A continuous function f from $[0, \infty)$ into itself is said to be operator monotone on $[0, \infty)$ if for two positive semidefinite operators A and B, the inequality $A \geq B$ implies $f(A) \geq f(B)$. An operator monotone function f on $[0, \infty)$ is said to be normal if $f(1) = 1$. Throughout this paper, OM_+^1 stands for the set of normalized operator monotone functions from $[0, \infty)$ into itself. Let A, B be positive invertible operators on $\mathcal{H}$. An operator mean $A\sigma B$ associated with $f \in OM_+^1$ is defined as

$$A\sigma B = A^{\frac{1}{2}} f\left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right) A^{\frac{1}{2}}$$

and its properties were studied in [4]. In particular, the weighted geometric mean $\#_\alpha$ associated with $f(t) = t^\alpha$ $(0 \leq \alpha \leq 1)$ has often been studied separately from the point of view of functional analysis, statistics, quantum information, or other fields. Ando and Hiai [2] proved the following theorem concerning weighted geometric means: If A, B are positive invertible operators with $A\#_\alpha B \leq 1$, then $A^r \#_\alpha B^r \leq 1$ for all $r \geq 1$. This theorem is referred to as the Ando–Hiai inequality. In [5], Fujii and Kamei showed that the Ando–Hiai inequality implies the following essential part of the Furuta inequality:

$$A \geq B > 0 \quad \Rightarrow \quad B^p \#_{\frac{p-1}{p+r}} A^{-r} \left(= A^{-r} \#_{\frac{1+r}{p+r}} B^p\right) \leq B \quad (r, p \geq 1). \tag{1.3}$$

In the present paper, we study a way to construct the following Furuta-type inequality:

$$A \geq B > 0 \quad \Rightarrow \quad X^{-r} \sigma_{\varphi_r} A^{-r} \leq B \quad (r \geq 1), \tag{1.4}$$

where X is the positive solution of $h(X) = B^{-1}$ for some function h. From the following Ando–Hiai type inequality: