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A construction of regular magic squares of odd order

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ABSTRACT

A magic square is an $n \times n$ array of numbers whose rows, columns, and the two diagonals sum to μ. A regular magic square satisfies the condition that the entries symmetrically placed with respect to the center sum to $\frac{2\mu}{n}$. Using circulant matrices we describe a construction of regular classical magic squares that are nonsingular for all odd orders. A similar construction is given that produces regular classical magic squares that are singular for odd composite orders. This paper is an extension of [3].

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1. Introduction

A *magic square* M is an $n \times n$ matrix in which entries along each row, each column, the main diagonal, and the cross diagonal add to the same value μ called the *magic sum* of M. If the entries of M are integers from 1 through n^2 where each number appears once then $\mu = \frac{n(n^2+1)}{2}$ and M is called a *classical* magic square (or *natural* magic square).

A magic square $M = [m_{i,j}]$ is said to be *regular* (also called *associated* or *symmetrical*) if the sum of the entries $m_{i,j}$ and $m_{n+1-i,n+1-j}$ that are symmetrically placed across the center of the square is equal to the number $\frac{2\mu}{n}$. In the case of classical magic square this sum is $n^2 + 1$.

Dürer's magic square

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}$$

is an example of a regular magic square [7]. In [5] Mattingly proved that every even order regular magic square is singular (that is, determinant of the magic square is zero). In [4] Loly et al. found that not all of the 5×5 regular classical magic squares are nonsingular. In [3] an example of a 9×9 regular classical magic square that is singular is given.

As a result the question of when an odd order regular magic square is singular or nonsingular was addressed in [3]. A necessary and sufficient condition for an odd order regular magic square to be nonsingular was given. In addition a method to construct nonsingular regular classical magic squares using circulant matrices is given when the order of the magic square is an odd prime power [3].

In this paper we extend this construction method of regular classical magic squares to all odd orders. Moreover, we show that this construction method will produce a singular or nonsingular regular classical magic square based on the choice of the first row of the circulant matrix used in the construction.

2. A construction of regular magic squares

In this section we present the method of construction used in [3] to produce regular classical magic squares.

Let E denote the matrix of all 1's for its entries and e denote the column vector of all 1's. Since $Me = \mu e$ we observe that the magic sum μ is an eigenvalue of magic square M. The following theorem is found in [1].

Theorem 2.1. *If M is an $n \times n$ magic square and ρ is a complex number, then $M + \rho E$ has the same eigenvalues of M except that μ is replaced with $\mu + \rho n$.*