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The rank distance problem for pairs of matrices and a completion of quasi-regular matrix pencils

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A B S T R A C T

In this paper we give a complete description of the possible feedback invariants of a pair of matrices submitted to an additive perturbation of low rank. Also, we describe the possible Kronecker invariants of a quasi-regular matrix pencil with a prescribed quasi-regular subpencil. All the results are valid over arbitrary fields.

1. Introduction

The problem of low-rank perturbations consists of determining the possible invariants of a matrix $X+Y$ with a matrix X given and a matrix Y being of fixed or bounded rank. These problems have been well studied in the last decade. Due to various possibilities for matrices X and Y (the ring or field over which they are defined, their size and form,

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the invariants in question), the problem depends heavily on the chosen case. The large majority of the results study changes in the Jordan structure of a constant matrix X (or the invariant factors of a square polynomial matrix X or of the Weierstrass canonical form in the case X is a regular matrix pencil), see e.g. [24,29,31] and the references therein. When dealing with singular matrix pencils, only few results are known [7,17,32].

The approaches used can be divided into two groups: the generic and the general one. In the generic approach certain properties of the matrix $X + Y$ are described for all matrices Y from an open dense subset of all matrices of fixed rank, see for example [30].

In the general approach the possible invariants of $X + Y$ are studied when Y varies through the set of all matrices of a given rank. Due to generality of the problem, much less results are known in this case. These include [7,27–29,33].

In this paper we follow the general approach.

The low-rank perturbation problem can be rewritten as the rank distance problem. Namely, by writing $Y = (X + Y) - X$ the problem becomes to find two matrices with prescribed invariants, such that their difference is of fixed or bounded rank.

In the general approach, the most important results for the rank distance problem are the ones by R.C. Thompson [33], E.M. de Sá [27] and F.C. Silva [28,29]. In [27,28,33] the rank distance problem for matrices over a principal ideal domain in the case when the involved invariants are for the equivalence relation has been studied and solved. Also, in [29] the rank distance problem for square matrices over a field, when the involved invariants are for the similarity relation, has been solved.

In this paper we are interested in pairs of matrices and the corresponding rectangular matrices. Pairs of matrices play an important role in control theory. Let $\mathbb{F}$ be an arbitrary field. Let $M \in \mathbb{F}^{p\times p}$ and $N \in \mathbb{F}^{p\times m}$. Time-invariant linear control systems are described by the following equation

$$\dot{x}(t) = Mx(t) + Nu(t) \tag{1}$$

where $x(t) \in \mathbb{F}^p$ is the state and $u(t) \in \mathbb{F}^m$ is the input of the system. The feedback control of the linear system (1) is given by $u(t) = Fx(t)$, for some $F \in \mathbb{F}^{m\times p}$. The result is a closed-loop system

$$\dot{x}(t) = (M + NF)x(t). \tag{2}$$

Properties of the systems (1) and (2) (e.g. controllability of (1) or existence of control F such that the system (2) is stable etc.), turn out to depend only on the pair $(M, N) \in \mathbb{F}^{p\times p} \times \mathbb{F}^{p\times m}$. Hence, studying the properties of a linear system reduces to studying the properties of the corresponding matrix pair. For more details see e.g. [3,26,35].

Two pairs of matrices $(M, N) \in \mathbb{F}^{p\times p} \times \mathbb{F}^{p\times m}$ and $(\bar{M}, \bar{N}) \in \mathbb{F}^{p\times p} \times \mathbb{F}^{p\times m}$ are said to be *feedback equivalent* if there exist invertible matrices $P \in \mathbb{F}^{p\times p}$, $Q \in \mathbb{F}^{m\times m}$ and a matrix $R \in \mathbb{F}^{m\times p}$ such that

$$\bar{M} = PMP^{-1} + PNR, \qquad \bar{N} = PNQ.$$