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An evolution algebra in population genetics

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A B S T R A C T

We consider an evolution algebra which corresponds to a bisexual population with a set of females partitioned into finitely many different types and the males having only one type. We study basic properties of the algebra. This algebra is commutative (and hence flexible), not associative and not necessarily power-associative, in general. We prove that being alternative is equivalent to being associative. We find conditions to be an associative, a fourth power-associative, or a nilpotent algebra. We also prove that if the algebra is not alternative then to be power-associative is equivalent to be Jordan. Moreover it is not unital. In a general case, we describe the full set of idempotent elements and the full set of absolute nilpotent elements. The set of all operators of left (right) multiplications is described. Under some conditions it is proved that the corresponding algebra is centroidal. Moreover the classification of 2-dimensional and some 3-dimensional algebras are obtained.

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1. Introduction

Description of a sex linked inheritance with algebras involves overcoming the obstacle of asymmetry in the genetic inheritance rules. Inheritance which is not sex linked is symmetrical with respect to the sexes of the organisms [5], while sex linked inheritance is not (see [4,6]). The main problem for a given algebra of a sex linked population is to carefully examine how the basic algebraic model must be altered in order to compensate for this lack of symmetry in the genetic inheritance system. In [2], Etherington began the study of this kind of algebras with the simplest possible case.

Now the methods of mathematical genetics have become probability theory, stochastic processes, nonlinear differential and difference equations and non-associative algebras. The book [5] describes some mathematical methods of studying algebras of genetics. This book mainly considers a *free population*, which means random mating in the population. Evolution of a free population can be given by a dynamical system generated by a quadratic stochastic operator (QSO) and by an evolution algebra of a free population. In [5] an evolution algebra associated to the free population was introduced and, using this non-associative algebra, many results are obtained in explicit form, e.g., the explicit description of stationary quadratic operators, and the explicit solutions of a nonlinear evolutionary equation in the absence of selection, as well as general theorems on convergence to equilibrium in the presence of selection. In [3] some recently obtained results and also several open problems related to the theory of QSOs are discussed. See also [5] for more detailed theory of QSOs.

Recently in [4] an evolution algebra $\mathcal{B}$ was introduced identifying the coefficients of inheritance of a bisexual population as the structure constants of the algebra. The basic properties of the algebra are studied. Moreover a detailed analysis of a special case of the evolution algebra (of bisexual population in which type “1” of females and males have preference) was given. Since the structural constants of the algebra $\mathcal{B}$ are given by two cubic matrices, the study of this algebra is difficult. To avoid such difficulties we have to consider an algebra of bisexual population with a simplified form of matrices of structural constants. In this paper we consider a such simplified model of bisexual population and study corresponding evolution algebra.

The paper is organized as follows. In Section 2 we define our algebra as an evolution algebra which corresponds to a bisexual population with a set of females partitioned into finitely many different types and the males having only one type. Then we study basic properties (associativity, non-associativity, commutativity, power-associativity, nilpotency, unitality, etc.) of the algebra. Section 3 is devoted to subalgebras, absolute nilpotent elements and idempotent elements of the algebra. In Section 4 the set of all operators of left (right) multiplications is described. In Section 5, under some conditions, it is proved that the corresponding algebra is centroidal. The last section gives a classification of 2-dimensional and some 3-dimensional algebras.