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On extensions of free nilpotent Lie algebras of type 2

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ABSTRACT

In this paper, we study the structure of Lie algebras which have free t-nilpotent Lie algebras $\mathfrak{n}_{2,t}$ of type 2 as nilradical and give a detailed construction for them. We prove that the dimension of any Lie algebra $\mathfrak{g}$ of this class is $\dim \mathfrak{n}_{2,t} + k$. If $\mathfrak{g}$ is solvable, $k \leq 2$; otherwise, the Levi subalgebra of $\mathfrak{g}$ is $\mathfrak{sl}_2(\mathbb{K})$, the split simple 3-dimensional Lie algebra of 2×2 matrices of trace zero, and then $k \leq 4$. As an application of the main results we get the classification over algebraically closed fields of Lie algebras with nilradical $\mathfrak{n}_{2,1}$, $\mathfrak{n}_{2,2}$ and $\mathfrak{n}_{2,3}$.

1. Introduction

From Levi's Theorem, any finite-dimensional Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$, direct sum of vector spaces, where $\mathfrak{r}$ is the solvable radical (the maximal solvable ideal of $\mathfrak{g}$) and $\mathfrak{s}$ is a semisimple subalgebra called *Levi subalgebra* (or Levi factor). The Lie

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algebra $\mathfrak{g}$ is faithful if the adjoint representation of the subalgebra $\mathfrak{s}$ in $\mathfrak{r}$ is faithful (equivalently, $\mathfrak{g}$ contains no nonzero semisimple ideals). Note that any Lie algebra with radical $\mathfrak{r}$ can be decomposed into the direct sum (as ideals) of a semisimple Lie algebra and a faithful Lie algebra with the same radical. Let now consider the nilradical $\mathfrak{n}$ of $\mathfrak{g}$, i.e. the largest nilpotent ideal inside $\mathfrak{g}$. The nilradical is contained in the solvable radical and:

$$[\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{n}. \tag{1}$$

In other words, the action of $\mathfrak{g}$ on $\frac{\mathfrak{r}}{\mathfrak{n}}$ is trivial.

Concerning the problem of the classification of Lie algebras of a given radical, Malcev [12] (see also [17, Theorem 4.4, Section 4]) proved the following structure result:

Theorem 1.1. *Any faithful Lie algebra $\mathfrak{g}$ with solvable radical $\mathfrak{r}$ is isomorphic to a Lie algebra of the form $\mathfrak{s} \oplus_{id} \mathfrak{r}$ where $\mathfrak{s}$ is a semisimple subalgebra of a Levi subalgebra $\mathfrak{s}_0$ of the algebra of derivations of $\mathfrak{r}$, named* $\operatorname{Der} \mathfrak{r}$. *Moreover, given $\mathfrak{s}_1$ and $\mathfrak{s}_2$ semisimple subalgebras of $\mathfrak{s}_0$, the algebras $\mathfrak{s}_1 \oplus_{id} \mathfrak{r}$ and $\mathfrak{s}_2 \oplus_{id} \mathfrak{r}$ are isomorphic if and only if $\mathfrak{s}_2 = A\mathfrak{s}_1 A^{-1}$, where A is an automorphism of $\mathfrak{r}$.* □

The product in $\mathfrak{s} \oplus_{id} \mathfrak{r}$ is given by considering $\mathfrak{s}$ and $\mathfrak{r}$ as subalgebras and $[d, a] := d(a)$ for $d \in \mathfrak{s}$ and $a \in \mathfrak{r}$; since $\mathfrak{s} \subseteq \operatorname{Der} \mathfrak{r}$, $[d_1, d_2] = d_1 d_2 - d_2 d_1$ is the Lie bracket for $d_i \in \mathfrak{s}$.

The previous theorem reduces the problem of classifying Lie algebras with a given solvable radical $\mathfrak{r}$, to the analysis of derivations and automorphisms of the solvable Lie algebra $\mathfrak{r}$. The same argument is valid if we consider the problem of classifying Lie algebras with a given nilradical. According to [13], solvable Lie algebras can be classified through nilpotent ones and, from [19], any nilpotent Lie algebra is a quotient of a free nilpotent Lie algebra of the same type and nilindex. So, as the former problem reduces to the latter one, it seems to be quite natural to start studying the classification of Lie algebras with a given nilradical. This is the starting point in [2] where Malcev's decompositions of Lie algebras are studied, or in [16] where the problem of classifying Lie algebras whose radical is just the nilradical of a parabolic subalgebra of a semisimple Lie algebra is treated. In this paper we classify Lie algebras with nilradical a free nilpotent Lie algebra $\mathfrak{n}_{2,t}$ of type 2 and nilindex t (i.e. $\mathfrak{n}_{2,t}^{t+1} = 0$, $\mathfrak{n}_{2,t}^{t} \neq 0$ and $\dim \mathfrak{n}_{2,t} - \dim \mathfrak{n}_{2,t}^2 = 2$). Following [19], the Levi subalgebra of $\operatorname{Der} \mathfrak{n}_{2,t}$ is $\mathfrak{sl}_2(\mathbb{K})$, the split 3-dimensional simple Lie algebra of 2×2 matrices of trace zero. So, our main idea in this paper is to apply basic representation theory of $\mathfrak{sl}_2(\mathbb{K})$ to attack the problem. This technique has been used previously in [21] to get the classification of the 9-dimensional nonsolvable indecomposable Lie algebras.

This paper is organized as follows. Section 2 gives some basic definitions and facts on free nilpotent Lie algebras $\mathfrak{n}_{2,t}$ and includes a complete description of $\operatorname{Der} \mathfrak{n}_{2,t}$, the Lie algebra of derivations of $\mathfrak{n}_{2,t}$. Sections 3 and 4 deal with the structure and general construction of Lie algebras with nilradical $\mathfrak{n}_{2,t}$. It turns out that the solvable radical of