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A note on geometric bounds for eigenvalues

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ABSTRACT

This paper finds a new and improved geometric bound for the second largest eigenvalue of a random walk Markov chain on an undirected graph.

1. Introduction

Let X be a finite set and $P = (P_{i,j})$ a transition matrix for an irreducible and reversible Markov chain, with respect to the probability $\pi = (\pi_i)$ on X; i.e.

$$\sum_{i \in X} \pi_i = 1.$$

Then define the symmetric (due to reversibility) $Q = (Q_{i,j})$, via

$$Q_{i,j} = \pi_i P_{i,j} = \pi_j P_{j,i} \quad \text{for all } i, j \in X.$$

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An irreducible reversible Markov chain can be derived from an irreducible non-negative symmetric matrix $A = (a_{ij})$. Define

$$r_i = \sum_{j=1}^{n} a_{ij}, \qquad R = \operatorname{diag}(r_1, \dots, r_n) \quad \text{and} \quad P = R^{-1}A.$$

Then P is a stochastic matrix; i.e.

$$\sum_{j=1}^{n} P_{i,j} = 1 \quad \text{for all } i,$$

and if $\pi_i = r^{-1} r_i$ where

$$r = \sum_{i=1}^{n} r_i,$$

then it follows that $\pi_i P_{i,j} = P_{j,i}\pi_j$ for all i, j.

To derive an upper bound for β_1, the second largest eigenvalue of P, [1] introduce the notion of a set of paths Γ which joins each pair of distinct elements of X. Let γ_{ij} denote the path linking i and j, for which there is positive probability the chain can move along; the existence of which is guaranteed due to the irreducibility of the chain. So

$$\Gamma = \{\gamma_{ij} : \forall i \neq j\}.$$

Then [1] shows that

$$\beta_1 \leq 1 - \frac{1}{K},$$

where, for $e = (e^-, e^+)$ with $e^-, e^+ \in X$ and $e^- \neq e^+$,

$$K = \max_e \frac{\gamma(e)}{Q(e)},$$

$Q(e) = Q_{e^-,e^+}$, and

$$\gamma(e) = \sum_{\gamma_{ij} \ni e} |\gamma_{ij}| \pi_i \pi_j,$$

with $|\gamma_{ij}|$ denoting the length of the path γ_{ij}. The superiority of this bound over alternatives, such as Cheeger bounds, is shown in [2].

In particular, [1] consider the Markov chain which is a random walk on the undirected graph $G = (X, E)$, and G is connected, has no loops or multiple edges. Following the same notation as [1] we have