Linear Algebra and its Applications 457 (2014) 408–427

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The moduli space of 4-dimensional nilpotent complex associative algebras ☆

Alice Fialowski [a,*], Michael Penkava [b]

[a] Eötvös Loránd University, Budapest, Hungary
[b] University of Wisconsin-Eau Claire, Eau Claire, WI 54702-4004, United States

ARTICLE INFO

Article history:
Received 3 April 2014
Accepted 8 May 2014
Available online 20 June 2014
Submitted by V.V. Sergeichuk

MSC:
14D15
13D10
14B12
16S80
16E40
17B55
17B70

Keywords:
Versal deformations
Associative algebras
Bilinear forms
Extensions

ABSTRACT

In this paper, we study 4-dimensional nilpotent complex associative algebras. This is a continuation of the study of the moduli space of 4-dimensional algebras. The nonnilpotent algebras were analyzed in an earlier paper. Even though there are only 15 families of nilpotent 4-dimensional algebras, the complexity of their behavior warranted a separate study, which we give here.

☆ Research of the first author was partially supported by OTKA grant K77757 and the second author by grants from the University of Wisconsin-Eau Claire.

* Corresponding author.
E-mail addresses: fialowsk@cs.elte.hu (A. Fialowski), penkavmr@uwec.edu (M. Penkava).

http://dx.doi.org/10.1016/j.laa.2014.05.014

1. Construction of the algebras by extensions

The authors and collaborators have been carrying out a construction of moduli spaces of low dimensional complex and real Lie and associative algebras in a series of papers. Our method of constructing the moduli spaces of such algebras is based on the principle that algebras are either simple or can be constructed as extensions of lower dimensional algebras. There is a classical theory of extensions, which was developed by many contributors going back as early as the 1930s. In order to apply this theory, the authors interpreted the classical theory of extensions in the language of codifferentials [4], wherein we give a description of the theory of extensions of an algebra W by an algebra M. Consider the diagram

$$0 \to M \to V \to W \to 0$$

of associative $\mathbb{K}$-algebras, so that $V = M \oplus W$ as a $\mathbb{K}$-vector space, M is an ideal in the algebra V, and $W = V/M$ is the quotient algebra. Suppose that $\delta \in C^2(W)$ and $\mu \in C^2(M)$ represent the algebra structures on W and M respectively. We can view μ and δ as elements of $C^2(V)$. Let $T^n(V)$ be the n-th tensor power of V, so that $T^0(V) = \mathbb{K}$ and $T^{k+1}(V) = V \otimes T^n(V)$. Let $T^{k,l}$ be the subspace of $T^{k+l}(V)$ given recursively by

$$\begin{aligned} T^{0,0} &= \mathbb{K} \\ T^{k,l} &= M \otimes T^{k-1,l} \oplus V \otimes T^{k,l-1} \end{aligned}$$

Let $C^{k,l} = \operatorname{Hom}(T^{k,l}, M) \subseteq C^{k+l}(V)$. If we denote the algebra structure on V by d, we have

$$d = \delta + \mu + \lambda + \psi,$$

where $\lambda \in C^{1,1}$ and $\psi \in C^{0,2}$. Note that in this notation, $\mu \in C^{2,0}$. Then the condition that d is associative: $[d, d] = 0$ gives the following relations:

$$\begin{aligned} [\delta, \lambda] + \frac{1}{2}[\lambda, \lambda] + [\mu, \psi] &= 0, \quad \text{The Maurer–Cartan equation} \\ [\mu, \lambda] &= 0, \quad \text{The compatibility condition} \\ [\delta + \lambda, \psi] &= 0, \quad \text{The cocycle condition} \end{aligned}$$

Since μ is an algebra structure, $[\mu, \mu] = 0$, so if we define D_μ by $D_\mu(\varphi) = [\mu, \varphi]$, then $D_\mu^2 = 0$. Thus D_μ is a differential on $C(V)$. Moreover $D_\mu : C^{k,l} \to C^{k+1,l}$. Let

$$Z_\mu^{k,l} = \ker\bigl(D_\mu : C^{k,l} \to C^{k+1,l}\bigr), \quad \text{the } (k,l)\text{-cocycles}$$