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# The largest subsemilattices of the endomorphism monoid of an independence algebra

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ABSTRACT

An algebra $\mathbb{A}$ is said to be an independence algebra if it is a matroid algebra and every map $\alpha : X \to A$, defined on a basis $X$ of $\mathbb{A}$, can be extended to an endomorphism of $\mathbb{A}$. These algebras are particularly well-behaved generalizations of vector spaces, and hence they naturally appear in several branches of mathematics such as model theory, group theory, and semigroup theory.

It is well known that matroid algebras have a well-defined notion of dimension. Let $\mathbb{A}$ be any independence algebra of finite dimension $n$, with at least two elements. Denote by $\mathrm{End}(\mathbb{A})$ the monoid of endomorphisms of $\mathbb{A}$. We prove that a largest subsemilattice of $\mathrm{End}(\mathbb{A})$ has either $2^{n-1}$ elements (if the clone of $\mathbb{A}$ does not contain any constant operations) or $2^n$ elements (if the clone of $\mathbb{A}$ contains constant operations). As corollaries, we obtain formulas for the size of the largest subsemilattices of: some variants of the monoid of linear operators of a finite-dimensional vector space, the monoid of full transformations on a finite set $X$, the monoid of partial transformations on $X$, the monoid of endomorphisms of a free $G$-set with a finite set of free generators, among others.

The paper ends with a relatively large number of problems that might attract attention of experts in linear algebra,

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ring theory, extremal combinatorics, group theory, semigroup theory, universal algebraic geometry, and universal algebra.


## 1. Introduction

Let $\mathbb{A} = \langle A; F\rangle$ be an algebra (as understood in universal algebra [48]). We say that $\mathbb{A}$ is a *matroid algebra* if the closure operator *subalgebra generated by*, denoted $\langle\cdot\rangle$, satisfies the *exchange property*, that is, for all $X \subseteq A$ and $x, y \in A$,

$$x \in \langle X \cup \{y\}\rangle \quad \text{and} \quad x \notin X \quad \Rightarrow \quad y \in \langle X \cup \{x\}\rangle. \tag{1.1}$$

(In certain contexts of model theory, a closure system satisfying the exchange property is called a *pre-geometry*.) A set $X \subseteq A$ is said to be *independent* if $X$ has no redundant elements, that is, $X$ is a minimal generating set for the subalgebra it generates: for all $x \in X$, $x \notin \langle X \setminus \{x\}\rangle$.

By standard arguments in matroid theory, we know that a matroid algebra has a *basis* (an independent generating set), and all bases have the same cardinality; thus matroid algebras admit a notion of dimension, defined as the cardinality of one (and hence all) of its bases. An *independence algebra* is a matroid algebra satisfying the *extension property*, that is, every map $\alpha : X \to A$, defined on a basis $X$ for $\mathbb{A}$, can be extended to an endomorphism of $\mathbb{A}$. Examples of independence algebras are vector spaces, affine spaces (as defined below), unstructured sets, and free $G$-sets.

The class of independence algebras was introduced by Gould in 1995 [46]. Her motivation was to understand the properties shared by vector spaces and sets that result in similarities in the structure of their monoids of endomorphisms. As pointed out by Gould, this notion goes back to the 1960s, when the class of $v^*$-algebras was defined by Narkiewicz [55]. (The "$v$" in $v^*$-algebras stands for "vector" since the $v^*$-algebras were primarily seen as generalizations of the vector spaces.) In fact, the $v^*$-algebras can be defined as the matroid algebras with the extension property [56], just like independence algebras, but with a slight difference. In the context of independence algebras, the subalgebra generated by the empty set is the subalgebra generated by all nullary operations; while in $v^*$-algebras, it is the subalgebra generated by the images of all constant operations. The effect of this difference, so tiny that it has gone unnoticed by previous authors, is that there do exist $v^*$-algebras $\mathbb{A} = \langle A; F\rangle$ that are not independence algebras, namely exactly those for which:

1. $|A| \geq 2$,
2. $\mathbb{A}$ does not have any nullary operations,
3. every element of $A$ is the image of some constant operation from the clone of $\mathbb{A}$.