Linear Algebra and its Applications 458 (2014) 80–98

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Spectra of order-4 special magic squares

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ARTICLE INFO

Article history:
Received 13 December 2013
Accepted 21 May 2014
Available online 20 June 2014
Submitted by R. Brualdi

MSC:
00A08
11C20
15A04
15A18

Keywords:
Magic squares
Matrix algebra
Eigenvalues
Eigenvectors
SVD

ABSTRACT

A new parameterization for order-4 regular (or associative) magic square matrices leads to general formulas for their eigenvalues, eigenvectors, and singular value decomposition. Known transformations extend these results to order-4 pandiagonal and bent-diagonal magic squares. The effect of various transformations on the eigenvalues and singular values of these special magic squares is considered. Numerical examples are presented and numerical values are obtained from simple formulas for the eigenvalues and singular values of each of the 48 natural pandiagonal, regular, and bent-diagonal magic squares of order 4 and their reflections.

1. Introduction

Previously, Loly et al. [9] studied the spectra and singular value decomposition (SVD) of magic square matrices, including those of order 4, and gave numerical results for many cases. Some order-4, regular (or associative) magic squares were found to have three zero eigenvalues but their associated generalized eigenvectors were not obtained. Recently, Nordgren [12] studied the spectra and other properties of regular and pandiagonal magic

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http://dx.doi.org/10.1016/j.laa.2014.05.038

square matrices. Basic material on magic squares and references to numerous other works may be found in [9,15–17].

In the present paper, using a new parameterization for order-4 regular magic square matrices, we find general formulas for their eigenvalues, eigenvectors, and SVD, including the case of three zero eigenvalues. These formulas are extended to order-4 pandiagonal magic squares by the Planck transformation [18,12] and to order-4 bent-diagonal magic squares by a transformation due to Heinz [7] which we write in matrix form. We study several interesting transformations that follow from our parameterization. First, we verify that reflection of a regular magic square multiplies its signed pair of eigenvalues by the factor i as often observed for all orders and proved in [12]. Second, regular and pandiagonal magic squares are found to undergo spectrum-preserving transformations on exchange of certain elements. Third, transformations resulting from shifting the rows and columns of pandiagonal magic squares are studied and the effect of these transformations on their eigenvalues is determined. Transformations of these shifted squares extend these results to regular and bent-diagonal magic squares. We show that singular values of a magic square are invariant under row/column interchanges. Also, we verify that odd powers of regular magic squares are regular and odd powers of most-perfect magic squares are most perfect as proved in [12]. In addition, we find that odd powers of bent-diagonal magic squares are bent-diagonal.

The theoretical results presented here are illustrated with numerical examples. In particular, numerical values of the eigenvalues and singular values of each of the known 48 order-4 natural pandiagonal, bent-diagonal, and regular magic squares (Dudeney [5] groups I, II, III) and their reflections are obtained from simple formulas and tabulated values for each of the three sets of 16 squares. These numerical results are in general agreement with values reported by Loly et al. [9] for group III. However, comparison of eigenvalues for specific squares is not possible except for 16 specific squares identified in [9].

2. Regular magic squares

Parameterization. By definition, the rows and columns of a *semi-magic square* sum to the *index* m. The main diagonal and cross diagonal of a *magic square* also sum to m. In an order-4, *regular* magic (or *regmagic*) square, all pairs of elements that are symmetric with respect to its center sum to $m/2$. These conditions can be expressed as

$$M_R E = E M_R = mE, \qquad M_R + R M_R R = \frac{m}{2} E, \tag{1}$$

where

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2}$$

The diagonal sum conditions of a regmagic square follow from (1).