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A note on matrices with prescribed off-diagonal submatrix and characteristic polynomial ☆

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ABSTRACT

Let $A \in \mathbb{F}^{q\times p}$ be a nonzero matrix over a field $\mathbb{F}$, and let f be any monic polynomial of degree $n = p+q$ with coefficients in $\mathbb{F}$. A completion problem proposed and solved by de Oliveira asks for the existence of a matrix of order n with characteristic polynomial f and such that its $q \times p$ submatrix in the left down corner is equal to A. Let $C(f) \in \mathbb{F}^{n\times n}$ be the companion matrix of f. We will construct a matrix $X_A \in \mathbb{F}^{n\times n}$ (which depends exclusively on A) such that $X_A^{-1}C(f)X_A$ is a closed form solution of this completion problem.

1. Introduction

Let $\mathbb{F}$ be a field and consider the 2×2 block matrix

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad \text{with } M_{11} \in \mathbb{F}^{p\times p} \text{ and } M_{22} \in \mathbb{F}^{q\times q}. \tag{1}$$

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In the seventies de Oliveira posed the problem of determining all possible spectra or characteristic polynomials or invariant polynomials of $[M_{ij}]_{i,j=1}^2$ when some of the blocks are prescribed and the rest of the blocks are free to take any value. Summaries of solved cases can be found in [1,2,4].

In this work we are interested in the particular problem of determining all possible characteristic polynomials of $[M_{ij}]_{i,j=1}^2$ when only M_{21} is prescribed. Note that if M_{21} is the zero matrix then the solution is trivial, namely, a polynomial f of degree $p+q$ is a possible characteristic polynomial if and only if f has a divisor of degree p.

On the other hand, if M_{21} is a nonzero matrix then de Oliveira [3] proved that for any polynomial f of degree $p+q$ there exist some $M_{11} \in \mathbb{F}^{p\times p}$, $M_{12} \in \mathbb{F}^{p\times q}$ and $M_{22} \in \mathbb{F}^{q\times q}$ such that f is the characteristic polynomial of $[M_{ij}]_{i,j=1}^2$. And Ikramov and Chugunov [4] obtained a finite step algorithm to construct concrete M_{11}, M_{12} and M_{22}. In Lemma 1.1 and Theorem 1.1 below we develop a construction that provides a closed form solution. Before we introduce some useful notation.

Definition 1.1. We say that P is a **partially prescribed** matrix over a field $\mathbb{F}$ if some of the entries of P are known elements of $\mathbb{F}$ and the rest of the entries of P are unknown. Each unknown entry of P will be denoted by $\square$. A **completion** of P is any matrix obtained from P by replacing each unknown entry $\square$ by an element of $\mathbb{F}$.

For instance, if

$$P = \begin{bmatrix} 3 & \square & 5 \\ \square & \square & 7 \\ 2 & \square & \square \end{bmatrix}, \qquad Q = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 7 \\ 2 & 9 & 6 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 3 & -3 & 5 \\ 1 & 2 & 7 \\ 2 & 8 & 4 \end{bmatrix}$$

then Q and R are two different completions of the partially prescribed matrix P.

Lemma 1.1. *Let $\mathbb{F}$ be a field and consider the $(p+q)\times(p+q)$ partially prescribed matrix P over $\mathbb{F}$ which is given by*

$$P = \begin{bmatrix} \square & \square \\ A & \square \end{bmatrix} = \left[\begin{array}{ccc|ccc} \square & \cdots & \square & \square & \cdots & \square \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ \square & \cdots & \square & \square & \cdots & \square \\ \hline a_{11} & \cdots & a_{1p} & \square & \cdots & \square \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{q1} & \cdots & a_{qp} & \square & \cdots & \square \end{array}\right] \quad \textit{where } A \in \mathbb{F}^{q\times p} \textit{ and } a_{1p} = 1, \qquad (2)$$

and let