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A Putnam–Fuglede commutativity property for Hilbert space operators

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Given Hilbert space operators $A, B \in B(\mathcal{H})$, define $\delta_{A,B}$ and $\triangle_{A,B}$ in $B(B(\mathcal{H}))$ by $\delta_{A,B}(X) = AX - XB$ and $\triangle_{A,B}(X) = AXB - X$ for each $X \in B(\mathcal{H})$. An operator $A \in B(\mathcal{H})$ satisfies the Putnam–Fuglede properties δ, respectively $\triangle$ (notation: $A \in \mathrm{PF}(\delta)$, respectively $A \in \mathrm{PF}(\triangle)$), if for every isometry $V \in B(\mathcal{H})$ for which the equation $\delta_{A,V^*}(X) = 0$, respectively $\triangle_{A,V^*}(X) = 0$, has a non-trivial solution $X \in B(\mathcal{H})$, the solution X also satisfies $\delta_{A^*,V}(X) = 0$, respectively $\triangle_{A^*,V}(X) = 0$. We prove that an operator $A \in B(\mathcal{H})$ is in $\mathrm{PF}(\triangle)$ if and only if it is in $\mathrm{PF}(\delta)$.

1. Introduction

Given Hilbert space operators $A, B \in B(\mathcal{H})$, let $\delta_{A,B}$ and $\triangle_{A,B} \in B(B(\mathcal{H}))$ denote, respectively, the generalized derivation $\delta_{A,B}(X) = AX - XB$ and the elementary operator $\triangle_{A,B}(X) = AXB - X$ for each $X \in B(\mathcal{H})$. The classical Putnam–Fuglede

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commutativity theorem says that if A, B are normal (see, e.g., [7, p. 84]), then $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$. A similar result holds for $\triangle_{A,B}$ [12, Theorem 5(i)]: if A, B are normal, then $\triangle^{-1}_{A,B}(0) \subseteq \triangle^{-1}_{A^*,B^*}(0)$. These are symmetric versions of the Putnam–Fuglede commutativity theorem, which, in general, fail to extend to classes of Hilbert space operators more general than the class of normal operators. For instance, if A and B are subnormal operators, then $\delta_{A,B}(X) = 0$ for some $X \in B(\mathcal{H})$ does not always imply $\delta_{A^*,B^*}(X) = 0$. An asymmetric version of the commutativity theorems, wherein one replaces the pair of operators $\{A, B\}$ by the pair $\{A, B^*\}$, is known to hold for A and B^* belonging to a number of classes of operators which properly contain the class of normal operators. For example, [13,2], if A is dominant and B^* is hyponormal (i.e., if B is cohyponormal), then $\delta^{-1}_{A,B}(0) \subseteq \delta^{-1}_{A^*,B^*}(0)$.

We say that an operator $A \in B(\mathcal{H})$ *satisfies the Putnam–Fuglede property* δ (*resp.*, $\triangle$), $A \in \mathrm{PF}(\delta)$ (*resp.*, $A \in \mathrm{PF}(\triangle)$), *if* (*either trivially* A *is unitary or*) *for every isometry* $V \in B(\mathcal{H})$ *for which the equation* $\delta_{A,V^*}(X) = 0$ (*resp.*, $\triangle_{A,V^*}(X) = 0$) *has a non-trivial solution* $X \in B(\mathcal{H})$, *the solution* X *also satisfies* $\delta_{A^*,V}(X) = 0$ (*resp.*, $\triangle_{A^*,V}(X) = 0$). Property $\mathrm{PF}(\delta)$ gives rise to a characterization of contractions A with $C_{\cdot 0}$ completely non-unitary (shortened, henceforth, to cnu) part [4] (also see [3]): *A contraction* $A \in B(\mathcal{H})$ *has* $C_{\cdot 0}$ *cnu part if and only if* $A \in \mathrm{PF}(\delta)$.

A *part of an operator* $A \in B(\mathcal{H})$ is its restriction to a closed invariant subspace, and we say that a part $A|_M$ *of* A *is reducing if the* (*closed invariant*) *subspace* M *reduces* A. A well-known basic result for Hilbert space contractions reads as follows (see, e.g., [7, Problem 4.10]). If $A \in B(\mathcal{H})$ is a contraction, then every unitary part of A (i.e., every restriction of A to a closed invariant subspace M of A such that $A|_M$ is unitary) is reducing. In other words, *if the restriction of a Hilbert space contraction to an invariant subspace is unitary, the subspace is reducing.* This, however, fails for a general operator $A \in B(\mathcal{H})$.

The purpose of this note is to investigate Putnam–Fuglede properties (shortened to PF-properties) for operators acting on Hilbert spaces. We prove that an operator $A \in B(\mathcal{H})$ satisfies property $\mathrm{PF}(\delta)$ if and only if it satisfies $\mathrm{PF}(\triangle)$. This, by [11, Theorem 3.2], then implies that a power bounded operator $A \in \mathrm{PF}(\triangle)$ if and only if A is the direct sum of a unitary with a $C_{\cdot 0}$ operator. For operators $A \in B(\mathcal{H})$ such that $A^{-1}(0) \subseteq A^{*-1}(0)$, a necessary and sufficient condition for $A \in \mathrm{PF}(\delta)$ is that the unitary parts of A reduce A. The requirement that $A^{-1}(0) \subseteq A^{*-1}(0)$ is however not necessary (as follows from the fact that every contraction in $B(\mathcal{H})$ with a $C_{\cdot 0}$ cnu part satisfies property $\mathrm{PF}(\delta)$). We point out that a number of the more commonly considered classes of Hilbert space operators, amongst them hyponormal, p-hyponormal, dominant and k-$*$-paranormal operators, satisfy the PF-property.

2. Main results

Given an operator $X \in \delta^{-1}_{A,V^*}(0)$ for some operator A and isometry V in $B(\mathcal{H})$, it is clear that $\overline{\mathrm{ran(X)}}$ is invariant under A and $\ker(X)^{\perp}$ is invariant under V: We shall