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On the term rank partition ☆

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For each positive integer t, the t-term rank of a $(0,1)$-matrix A is the maximum number of 1's in A with at most one 1 in each column and at most t 1's in each row. In [5] R. Brualdi et al. (2012) stated several results for the t-term rank, including a formula for the maximum t-term rank over a nonempty class of $(0,1)$-matrices with the same row sum and column sum vectors. In this paper we state more results for the t-term rank. Using these results we define and we study the term rank partition. We also deduce a formula for the minimal t-term rank over a nonempty class of $(0,1)$-matrices with the same row sum and column sum vectors.

1. Introduction

Let A be an m-by-n, $(0,1)$-matrix. Let t be a positive integer. Motivated by the study of combinatorial batch codes (see [4,7,9]), the authors of [5] defined the *t-term rank* of A,

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denoted $\rho_t(A)$, as the maximum number of 1's in A with at most one 1 in each column and at most t 1's in each row.

When $t = 1$, we have the well-known term rank of A, denoted $\rho(A)$. By the König–Egerváry theorem (see [3], p. 6) $\rho(A)$ equals the minimum number of lines that cover all the 1's of A:

$$\rho(A) = \min\{e + f : \exists \text{ a cover of } A \text{ with } e \text{ rows and } f \text{ columns}\}.$$

In [5] the authors established a generalization of this theorem:

Proposition 1.1. *Let A be an m-by-n, $(0,1)$-matrix and let t be a positive integer. Then*

$$\rho_t(A) = \min\{te + f : \exists \textit{ a cover of } A \textit{ with } e \textit{ rows and } f \textit{ columns}\}. \qquad \square$$

Let L be a subset of $\{1, \ldots, m\}$ and H be a subset of $\{1, \ldots, n\}$. We denote by $A[L; H]$ the $|L|$-by-$|H|$ submatrix of A determined by the rows with index in L and the columns with index in H. If $L = \{a_1, \ldots, a_p\}$ and $H = \{b_1, \ldots, b_v\}$ then, sometimes we will write $A[a_1, \ldots, a_p; b_1, \ldots, b_v]$ instead of $A[L; H]$. When $L = \{1, \ldots, m\}$, we write $A[\star; H]$ instead of $A[L; H]$.

The t-term rank of A can be obtained using the rank of the matroid union (see [1,5,6,8]) of t copies of $M(A)$, the matroid on $\{1, 2, \ldots, n\}$ (n is the number of columns of A), where the independent sets of $M(A)$ are those $K \subseteq \{1, \ldots, n\}$ such that, the m-by-$|K|$ submatrix of A, $A[*; K]$, verify $\rho(A[*; K]) = |K|$. Such matroids are transversal matroids and the rank of $M(A)$ is $\rho(A)$. The matroid union of t copies of $M(A)$, this is, the transversal matroid $M(A^{(t)})$ with

$$M\big(A^{(t)}\big) = M(A) \vee \ldots \vee M(A) \quad \big(t \text{ copies of } M(A)\big),$$

has rank equal to $\rho_t(A)$ and its bases (maximal independent sets) are those $K \subseteq \{1, \ldots, n\}$ such that $|K| = \rho_t(A[*; K]) = \rho_t(A)$.

Let l be a positive integer. A partition of weight l is a sequence of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_u)$ satisfying

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_u \geq 0$$

and

$$\sum_{i=1}^{u} \alpha_i = l.$$

If $\alpha = (\alpha_1, \ldots, \alpha_p)$ is a partition of weight l and $\alpha_p > 0$ we say that p is the length of α (that is, the length of α is the greatest k such that $\alpha_k > 0$).