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The spectral connection matrix for classical orthogonal polynomials of a single parameter

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ABSTRACT

In this paper we study the so-called connection problem of, given a polynomial expressed in the basis of one set of orthogonal polynomials, computing the coefficients with respect to a different set of orthogonal polynomials. We restrict our current study to the classical real orthogonal polynomials of the Hermite, Laguerre, and Gegenbauer (including Legendre) families.

The computational tool for this work is the class of quasiseparable matrices. While the relationships between orthogonal polynomials and rank-structured matrices such as quasiseparable matrices are very well-known, in this paper we investigate a more recently considered relationship. We prove that, while the matrix that implements the desired connection is not itself quasiseparable, it is an eigenvector matrix of one that is quasiseparable. We suggest to refer to this structured matrix as the spectral connection matrix.

Finally, we present a simple algorithm exploiting the computationally favorable properties of quasiseparable matrices to implement the desired change of basis. By exploiting the quasiseparable structure, this algorithm enjoys an order of magnitude reduction of complexity as compared to the simple method of inverting the connection matrix directly. While not the focus of the paper, some very preliminary numerical experimentation shows some positive indications that even with this reduction in complexity the accuracy of the resulting change of basis algorithm is comparable to that of inverting the connection matrix directly.

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1. Introduction

Let $\{P_k(x)\}_{k=0}^{\infty}$ be a sequence of real-valued polynomials, with $\deg(P_k(x)) = k$, and let $w(x)$ be a non-negative real-valued weight function on some interval $[a, b]$. Then $\{P_k(x)\}_{k=0}^{\infty}$ is said to be orthogonal with respect to the weight function $w(x)$ on $[a, b]$ if for each $j \neq k$, $\langle P_k, P_j \rangle = \int_a^b P_k(x)P_j(x)w(x)\,dx = 0$. The expansions of polynomials in bases of such orthogonal polynomials are of interest in mathematics, among many other uses of orthogonal polynomials. In this paper we will be concerned specifically with the Hermite, Laguerre, and Gegenbauer families, which are classical real orthogonal polynomials defined by (at most) a single parameter. While the Gegenbauer families compose the subset of Jacobi polynomials with two equal parameters, it should be noted that the general Jacobi case is not considered here, as in general Jacobi polynomials are described by two parameters, namely α and β describing the weight function $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$. Extensions to include this and other cases are the topic of forthcoming work.

The classical orthogonal polynomials are useful in applications too numerous to include a full list, but notably include Gaussian quadrature [15], random matrix theory [13], fluid dynamics [29], and computations in quantum mechanics [32]; for a longer attempt at listing applications, see [24].

We consider the problem of, given constants $\{a_k\}$ and orthogonal polynomials $\{P_k\}$ and $\{Q_k\}$, computing constants $\{b_k\}$ such that $\sum a_k P_k = \sum b_k Q_k$. We'll refer to this as the connection problem, and further distinguish between the cases where $\{P_k\}$ and $\{Q_k\}$ belong to the same family of orthogonal polynomials (both are Gegenbauer for different values of the defining parameter, etc.) and where they do not. The simpler former case can be considered as a change of parameter, while the latter involves changing between different families of orthogonal polynomials. In this paper we restrict attention to the connection problem within and between the classical real orthogonal polynomials defined by a single parameter, listed above. The connection problem appears in areas such as harmonic analysis [38], mathematical physics [3], combinatorics [35], etc. There has been particular interest in the positivity of connection coefficients as well [14,36,37,40].

It is obvious that the connection problem can be solved by determining the entries of a suitable connection matrix. While the recurrence relations satisfied by real orthogonal polynomials may seem to reduce the computational complexity of this approach, inversion at a cost of $\mathcal{O}(n^3)$ operations is still required. The entries of this connection matrix itself, commonly called the connection coefficients, also have applications in pure mathematics, applied mathematics, and physics, as is studied in [2,3,14,35,37].