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Bruhat order of tournaments

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ABSTRACT

We extend the Bruhat order on the set $\mathcal{S}_n$ of permutations (permutation matrices) of $\{1, 2, \ldots, n\}$ and its generalization to classes $\mathcal{A}(R, S)$ of (0, 1)-matrices with row sum vector R and column sum vector S, to a Bruhat order on classes $\mathcal{T}(R)$ of tournaments with score vector R. As in the case of the Bruhat order on $\mathcal{A}(R, S)$, there are two possible Bruhat orders where one is a refinement of the other. We characterize the cover relation for one of these orders. For a special family of score vectors, we show these Bruhat orders are isomorphic to the partially ordered set of all subsets of a set.

1. Introduction

Denote by K_n the complete graph with vertices $\{1, 2, \ldots, n\}$. A *tournament* of order n is an orientation of K_n. A *tournament matrix* $T = [t_{ij}]$ is the $n \times n$ adjacency matrix

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of a tournament of order n. Thus T is a $(0,1)$-matrix such that $T + T^t = J_n - I_n$ where J_n is the $n \times n$ matrix of all 1s. In general, we shall not distinguish between a tournament and a tournament matrix and usually refer to both as tournaments and label both as T. The adjacency matrix of a tournament presupposes some linear listing of the vertices. Changing the order of the vertices replaces T with PTP^t for some permutation matrix P. The *score vector* of T is $R = (r_1, r_2, \ldots, r_n)$ where r_i is the number of 1s in row i, that is, the ith row sum of T. The score vector of T is the vector of outdegrees of the vertices of T. The outdegrees determine the indegrees, since the sum of the outdegree and indegree of a vertex is $n-1$. The well-known theorem of Landau [12] asserts that a vector $R = (r_1, r_2, \ldots, r_n)$ of nonnegative integers is the score vector of a tournament of order n if and only if

$$\sum_{i \in J} r_i \geq \binom{|J|}{2} \ \big(J \subseteq \{1, 2, \ldots, n\}\big), \quad \text{with equality if } J = \{1, 2, \ldots, n\}. \tag{1}$$

If $r_1 \leq r_2 \leq \cdots \leq r_n$, which can be assumed after a reordering of the vertices, then (1) is equivalent to

$$\sum_{i=1}^{k} r_i \geq \binom{k}{2} \ (k = 1, 2, \ldots, n), \quad \text{with equality if } k = n. \tag{2}$$

There are many proofs of Landau's theorem available; see, for instance, [1,5,11,13,14,16].

Let $\mathcal{T}(R)$ denote the set of all tournaments with score vector R. We introduce a partial order on the set $\mathcal{T}(R)$ which is motivated by the classical Bruhat order on the set $\mathcal{S}_n$ of permutations of $\{1, 2, \ldots, n\}$, which we regard as $n \times n$ permutation matrices. In [4,2] a Bruhat order was extended to sets $\mathcal{A}(R, S)$ of all $m \times n$ $(0,1)$-matrices with a specified row sum vector R and column sum vector S. If $m = n$ and $R = S = (1, 1, \ldots, 1)$, then $\mathcal{A}(R, S) = \mathcal{S}_n$, and the Bruhat orders coincide. This partial order is defined as follows. For an $m \times n$ $(0,1)$-matrix $X = [x_{ij}]$, the *leading partial sum matrix* of X is the $m \times n$ nonnegative integral matrix

$$\Sigma(X) = [\sigma_{ij}] \quad \text{where } \sigma_{ij} = \sigma_{ij}(A) = \sum_{k \leq i, l \leq j} x_{kl} \ (1 \leq i \leq m, 1 \leq j \leq n).$$

Note that if the row sum vector of X is $R = (r_1, r_2, \ldots, r_m)$, then

$$\sigma_{in} = r_1 + r_2 + \cdots + r_i \quad (i = 1, 2, \ldots, n).$$

If A and A' are matrices in $\mathcal{A}(R, S)$, then A *precedes* A' *in the Bruhat order*, written as $A \preceq_B A'$ provided that

$$\Sigma(A) \geq \Sigma\big(A'\big) \quad \text{(entrywise order)}.$$