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Distance spectra of graphs: A survey

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ABSTRACT

In 1971, Graham and Pollack established a relationship between the number of negative eigenvalues of the distance matrix and the addressing problem in data communication systems. They also proved that the determinant of the distance matrix of a tree is a function of the number of vertices only. Since then several mathematicians were interested in studying the spectral properties of the distance matrix of a connected graph. Computing the distance characteristic polynomial and its coefficients was the first research subject of interest. Thereafter, the eigenvalues attracted much more attention. In the present paper, we report on the results related to the distance matrix of a graph and its spectral properties.

1. Introduction

There are mainly two versions of the distance matrix of a graph: *graph-theoretical* and *geometric*. For a connected graph, the distance matrix, in the case of graph-theoretical version, is a natural generalization, with more specificity, of the adjacency matrix. The distance between two vertices is defined as the length (number of edges) of a shortest path between them. In the case of the geometric version, we consider points in a plane and

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the distances correspond to the Euclidean distance. In this case, we speak more about points in a *metric space* than about vertices in a graph. The origins of the distance matrix goes back to the very first paper of Cayley [34] in 1841. However, its study began formally during the 20-th century [120,146]. Graph theory researchers were first interested in the problem of realizability of the distance matrix. Namely, for a given real symmetric $n \times n$-matrix $D = (d_{i,j})$ such that $d_{i,i} = 0$ and $0 \leq d_{i,j} \leq d_{i,k} + d_{k,j}$, $1 \leq i, j, k \leq n$, is there a graph G for which D is the distance matrix. This problem was first posed by Hakimi and Yau [70], and then studied by many mathematicians among which we cite Simões-Pereira [122,124,126,127], Buneman [28], Simões-Pereira and Zamfirescu [128], Varone [140], Boesch [18], Patrinos and Hakimi [109], Bandelt [13], and Nieminen [106]. Dress [52] proved that any shortest path distance matrix can be realized by a minimum weight graph. In the case of trees, efficient algorithms stating how to find this optimal solution have been developed [18,40,109,128]. However, in the case of general graphs, only a few results are known concerning the structure of optimal realizations (see *e.g.* [53,73,81,90]), and dealing with the problem is much harder. Indeed, although it is well-known that an optimal realization exists [52,125], Althöfer [1] and Winkler [143] showed that the problem is NP-complete if the distance matrix has integer entries. Actually, many heuristic methods were proposed [53,73,106,122,123,128,139], however, computing optimal realizations of general distance matrices is still difficult.

The second aspect of distance matrix that kept the attention of the mathematicians is the study of its spectral properties. In this case, the focus was more on the graph theoretical version of the matrix. The interest began during the 70's with the appearance of the paper [65] by Graham and Pollack. In that paper the authors established a relationship between the number of negative eigenvalues of the distance matrix and the addressing problem in data communication systems. In the same paper [65], it was proved that the determinant of the distance matrix of a tree is a function of the number of vertices only. This impressive result made distance matrix spectral properties a research subject of great interest. Graham and Lovász [63] computed the inverse of the distance matrix of a tree. Edelberg, Garey and Graham [56], Graham and Lovász [63], and Hosoya, Murakami and Gotoh [75] studied the characteristic polynomial. Actually, they calculated certain, and in some cases all, the coefficients of the distance characteristic polynomial. Merris [102] provided the first estimation of the distance spectrum of a tree. Many other authors studied the distance spectrum of a graph, we report about their works below. Recently, the maximum or the minimum values of the distance spectral radius of a given class of graphs has been studied extensively.

Several domains of application of the distance matrix, in an implicit or an explicit form, are known: the design of communication networks [57,65], network flow algorithms [51,60], graph embedding theory [49,56,63,64,66] as well as molecular stability [75,159]. Balaban, Ciubotariu and Medeleanu [9] proposed the use of the distance spectral radius as a molecular descriptor (see also [39,137]). Gutman and Medeleanu [69] used the distance spectral radius to infer the extent of branching and model boiling points of an alkane (see also [19]). For other applications in chemistry see [74,103,115–117] as well as