Linear Algebra and its Applications 458 (2014) 387–402

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Sharp covering of a module by cyclic submodules

I.N. Nakaoka [a,*,1], E.L. Monte Carmelo [a,2], O.J.N.T.N. dos Santos [b]

[a] Departamento de Matemática, Universidade Estadual de Maringá, Brazil
[b] Universidade Estadual de Mato Grosso do Sul, Dourados, Brazil

A R T I C L E   I N F O

Article history:
Received 25 February 2013
Accepted 10 June 2014
Available online 27 June 2014
Submitted by R. Brualdi

MSC:
13M99
05E20
51E20
94B65

Keywords:
Covering
Module
Linear dependence
Action of group
Min–max principle

A B S T R A C T

Let A be a finite commutative ring with identity. A subset H of the A-module A^n is called an R-short covering of A^n if every element of this module can be written as a sum of a multiple of an element in H and an A-linear combination with at most R canonical vectors. Let $c(A, n, R)$ be the minimum cardinality of an R-short covering of A^n. In this work, the numbers $c(A, n, 0)$ are computed when A is a direct product of chain rings (extending previous results by Yildiz et al.) and when A is a finite local ring such that $D(A)^2 = \{0\}$, where $D(A)$ denotes the set of all zero divisors of A. In order to obtain these results, we develop a method based on action of group, the min–max principle and pairwise weakly linearly independent sets (a concept introduced in this article). A structural connection between classical covering and short covering is described too. Thus a combination of previous results is applied to improve known bounds on short coverings for several values.

* Corresponding author.
E-mail addresses: innakaoka@uem.br (I.N. Nakaoka), elmcarmelo@uem.br (E.L. Monte Carmelo), ojneto@uems.br (O.J.N.T.N. dos Santos).

[1] The author was partially supported by Fundação Araucária, grant 18077 (14/2009).
[2] The author was partially supported by CNPq/MCT, grants 304635/2010-3 and 311518/2013-3.

http://dx.doi.org/10.1016/j.laa.2014.06.019
0024-3795/

1. Introduction

1.1. Preliminary problems

A collection of subgroups $\{G_1, \dots, G_k\}$ is a *group partition* of the group G if G is the union of these subgroups and each pairwise intersection $G_i \cap G_j$ contains only the null element of G. Necessary and sufficient conditions for the existence of a group partition have been investigated since the seminal paper by Miller [11], who proved that if an abelian group has a non-trivial partition, then the group must be an elementary abelian p-group. Several works about this topic are collected in [19].

An extension arises naturally in the context of vector space. Let $\mathbb{F}_q^n$ denote the n-dimensional vector space over the finite field $\mathbb{F}_q$. A family $\{V_i, \dots, V_k\}$ of subspaces of $\mathbb{F}_q^n$ is called a *vector partition* of $\mathbb{F}_q^n$ if it satisfies: $\mathbb{F}_q^n = \bigcup_{i=1}^k V_i$ (covering property) and $V_i \cap V_j = \{0\}$ for every $i \neq j$ (packing property). The literature on vector partition interplays theoretical results and applications. Herzog and Schönheim [6] established a connection between vector partitions and single-error-correcting perfect mixed linear codes. Besides coding theory, the partition problem can be applied to several geometric and combinatorial structures: partial spreads, uniformly resolvable designs, see [15] and its references. Let $\sigma_q(n,t)$ denote the minimum number of a vector partition of $\mathbb{F}_q^n$ in which the largest partition has dimension t, see [5]. For our purpose, we mention $\sigma_q(n,1) = (q^n - 1)/(q-1)$.

1.2. Formulation of the problem

This work deals with an extension of vector partition to a module over a finite ring. In view of Miller's result, there is not hope of finding a concept of partition for a general module, because even the principal submodules can have non-trivial intersection. However, we restrict our attention to the corresponding covering problem. Throughout this paper, all rings are assumed to be commutative with identity 1. Let A^n be the A-module in the usual way, where A denotes an arbitrary finite ring.

A subset H of A^n is an *R-short covering* of A^n if every element in A^n can be represented as a sum of a multiple of a vector in H and an A-linear combination of at most R canonical vectors of A^n, that is, for $v \in A^n$, there exists $h \in H$ and there exist scalars $\alpha, \alpha_1, \dots, \alpha_R \in A$ such that

$$v = \alpha h + \sum_{i=1}^{R} \alpha_i e_{j_i}, \tag{1.1}$$

where $\{e_1, \dots, e_n\}$ denotes the canonical base of A^n. The number $c(A,n,R)$ represents the minimum cardinality of an R-short covering of A^n.

Let us focus on the extremal case $R = 0$. A 0-short covering H of A^n can be regarded as a *cyclic covering* of A^n, since A^n becomes the union of cyclic submodules $[h] = Ah =$