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# Ordering trees and graphs with few cycles by algebraic connectivity

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ABSTRACT

Several approaches for ordering graphs by spectral parameters are presented in the literature. We can find graph orderings either by the greatest eigenvalue (spectral radius or index) or by the sum of the absolute values of the eigenvalues (the energy of a graph) or by the second smallest eigenvalue of the Laplacian matrix (the algebraic connectivity), among others. By considering the fact that the algebraic connectivity is related to the connectivity and shape of the graphs, several structural properties of graphs relative to this parameter have been studied. Hence, a large number of papers about ordering graphs by algebraic connectivity, mainly about trees and graphs with few cycles, have been published. This paper surveys the significant results concerning these topics, trying to focus on possible points to be investigated in order to understand the difficulties to obtain partial orderings via algebraic connectivity.



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## 1. Introduction

Ordering objects is a natural question and this is a particularly important problem in combinatorics. Accordingly, ordering of graphs is a particular class of problems that is extensively studied. Two obvious parameters, number of vertices and edges, are not enough to order graphs as there are too many objects with the same classification. What is usually understood by *ordering graphs* is to give a (total) order of all graphs with the same number of vertices, according to some invariant.

Most papers studying graph ordering use a *spectral* parameter to compare elements. For ordering results up to 1988, we refer to Cvetkovič et al. [7]. From that date, other results about the subject may be highlighted. For example, Hofmeister [29], Chang and Huang [6], Lin and Guo [41] and Li et al. [39] order trees by the largest eigenvalues of the adjacency matrices in a special class of trees. Zhang and Chen [71], Yu, Lu and Tang [69], Belardo, Marzi and Simiċ [3], Guo [21] and Li et al. [40] order graphs by adjacency and Laplacian spectral indices. Rada and Uzcàtegui [50] give an ordering of chemical trees by Randiċ index. Iliċ, in [30], Zhang et al. [74] and He and Le [27] investigate an ordering of trees by Laplacian coefficients. Xu [68] determines extremal trees relative to Harary indices. Wang [64] presents an order of Huckel trees according to minimal energies. Wang and Kang [65] and Guo [20] study ordering of graphs by energy and Laplacian energy. The Laplacian energy was also chosen for ordering trees by Trevisan et al. [63] and Fritscher et al. [18,17].

Among all papers dealing with this issue, several of them refer to the subject of ordering or finding extremal trees by algebraic connectivity. The first paper on this particular topic is due to Grone and Merris [25] and was published in 1990. After that, a great number of papers were published and, only in recent years, we can find around fifty references about the ordering of graphs via algebraic connectivity. For example, Zhang and Liu [73] order trees with nearly perfect matchings by algebraic connectivity. Biyikoglu and Leydold [4] investigate the structure of trees that have minimal algebraic connectivity among all trees with a given degree sequence. Shao et al. [58] order trees and connected graphs by algebraic connectivity and Li et al. [36] extend their results. Lal et al. [35] also study this invariant on unicyclic graphs and connected graphs with certain number of pendant vertices. For a given $n$, Wang et al. [66] determine the graph with the largest algebraic connectivity among graphs with diameter at most 4. Rojo et al. [54,55,2,56,57] investigate ordering of caterpillars, Rojo and Medina [53], and Rojo [51,52] study algebraic connectivity in the class of Bethe trees. Guo et al [24] order lollipop graphs. Liu and Liu [42] order unicyclic graphs with the smallest algebraic connectivity. Li et al. [38] order bicyclic graphs. Wang [67] classifies trees as a function of the algebraic connectivity in certain intervals. Zhang [72] studies algebraic connectivity in trees of diameter 4 and also the relation between diameter and this parameter. Yuan et al. [70] give classes of trees of largest algebraic connectivity.

As an attempt to sort the myriad of results and techniques available in the literature, our purpose in this work is to collect the most important results relative to ordering