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Resistance distance in subdivision-vertex join and subdivision-edge join of graphs ☆

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ABSTRACT

The subdivision graph $S(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. Let $G_1 \cup G_2$ be the disjoint union of two graphs G_1 and G_2. The subdivision-vertex join of G_1 and G_2, denoted by $G_1 \dot{\vee} G_2$, is the graph obtained from $S(G_1) \cup G_2$ by joining every vertex in $V(G_1)$ to every vertex in $V(G_2)$. The subdivision-edge join of G_1 and G_2, denoted by $G_1 \veebar G_2$, is the graph obtained from $S(G_1) \cup G_2$ by joining every vertex in $I(G_1)$ to every vertex in $V(G_2)$, where $I(G_1)$ is the set of inserted vertices of $S(G_1)$. In this paper, formulae for resistance distance in $G_1 \dot{\vee} G_2$ and $G_1 \veebar G_2$ are obtained.

1. Introduction

All graphs considered in this paper are simple and undirected. The resistance distance is a distance function on graphs introduced by Klein and Randić [7]. For two vertices u, v in a connected G, the *resistance distance* between u and v is defined to be the effective resistance between them when unit resistors are placed on every edge of G. Let $r_{uv}(G)$

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denote the resistance distance between u and v in G. Some results on resistance distance can be found in [2,3,5–7,9–11].

For a graph G, let A_G and B_G denote the adjacency matrix and vertex-edge incidence matrix of G, respectively. The matrix $L_G = D_G - A_G$ is called the *Laplacian matrix* of G, where D_G is the diagonal matrix of vertex degrees of G. For a matrix M, the $\{1\}$-*inverse* of M is a matrix X such that $MXM = M$. If M is singular, then it has infinite $\{1\}$-inverses. It is known that resistance distances in a connected graph G can be obtained from any $\{1\}$-inverse of L_G (see [1,2]).

The subdivision graph $\mathcal{S}(G)$ of a graph G is the graph obtained by inserting a new vertex into every edge of G. Let $G_1 \cup G_2$ be the disjoint union of two graphs G_1 and G_2. The *subdivision-vertex join* of G_1 and G_2, denoted by $G_1 \dot{\vee} G_2$, is the graph obtained from $S(G_1) \cup G_2$ by joining every vertex of $V(G_1)$ to every vertex of $V(G_2)$. The *subdivision-edge join* of G_1 and G_2, denoted by $G_1 \veebar G_2$, is the graph obtained from $S(G_1) \cup G_2$ by joining every vertex of $I(G_1)$ to every vertex of $V(G_2)$, where $I(G_1)$ is the set of inserted vertices of $\mathcal{S}(G_1)$ (see [8]). In this paper, formulae for resistance distance in $G_1 \dot{\vee} G_2$ and $G_1 \veebar G_2$ are obtained.

2. Preliminaries

For a square matrix M, the *group inverse* of M, denoted by $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$ and $MX = XM$. It is known that $M^{\#}$ exists if and only if $\operatorname{rank}(M) = \operatorname{rank}(M^2)$. If M is real symmetric, then $M^{\#}$ exists and $M^{\#}$ is a symmetric $\{1\}$-inverse of M. Actually, $M^{\#}$ is equal to the Moore–Penrose inverse of M since M is symmetric (see [4]).

We use $M^{(1)}$ to denote any $\{1\}$-inverse of a matrix M. Let $(M)_{uv}$ denote the (u, v)-entry of M.

Lemma 2.1. *(See [1,4].) Let G be a connected graph. Then*

$$r_{uv}(G) = \left(L_G^{(1)}\right)_{uu} + \left(L_G^{(1)}\right)_{vv} - \left(L_G^{(1)}\right)_{uv} - \left(L_G^{(1)}\right)_{vu} = \left(L_G^{\#}\right)_{uu} + \left(L_G^{\#}\right)_{vv} - 2\left(L_G^{\#}\right)_{uv}.$$

Let e denote the all-ones column vector.

Lemma 2.2. *Let S be a real symmetric matrix such that $Se = 0$. Then $S^{\#}e = 0$, $e^\top S^{\#} = 0$.*

Proof. Since $Se = 0$, we have $S^{\#}e = S^{\#}SS^{\#}e = (S^{\#})^2 Se = 0$, $e^\top S^{\#} = (S^{\#}e)^\top = 0$. □

Lemma 2.3. *Let $L = \begin{pmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{pmatrix}$ be the Laplacian matrix of a connected graph. If each column vector of $L_2^\top$ is $-e$ or a zero vector, then $N = \begin{pmatrix} L_1^{-1} & 0 \\ 0 & S^{\#} \end{pmatrix}$ is a symmetric $\{1\}$-inverse of L, where $S = L_3 - L_2^\top L_1^{-1} L_2$.*