Linear Algebra and its Applications 458 (2014) 512–525

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Lie n-derivations of unital algebras with idempotents

Yu Wang

Department of Mathematics, Shanghai Normal University, Shanghai 200234, China

ARTICLE INFO

Article history:
Received 4 May 2014
Accepted 17 June 2014
Available online 3 July 2014
Submitted by M. Bresar

MSC:
16W25

Keywords:
Lie n-derivation
Singular Jordan derivation
Antiderivation
Derivation
Unital algebra

ABSTRACT

Let $\mathcal{A}$ be a unital algebra with nontrivial idempotents. We show that under certain assumptions every Lie n-derivation φ on $\mathcal{A}$ is of the form $\varphi = d+\delta+\gamma$, where d is a derivation of $\mathcal{A}$, δ is both a singular Jordan derivation and an antiderivation of $\mathcal{A}$, and γ is a linear map from $\mathcal{A}$ to its center $Z(\mathcal{A})$ that vanishes on $[\cdots[[\mathcal{A},\mathcal{A}],\mathcal{A}],\dots,\mathcal{A}]$. As an application we give a description of Lie n-derivations of unital algebras with wide idempotents.

1. Introduction

Throughout the paper, by an algebra we shall mean an algebra over a fixed unital commutative ring R, and we assume without further mentioning that $\frac{1}{2} \in R$.

Let $\mathcal{A}$ be an algebra. Recall that a linear map $d : \mathcal{A} \to \mathcal{A}$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{A}$. Similarly, a linear map $d : \mathcal{A} \to \mathcal{A}$ is called an *antiderivation* if $d(xy) = d(y)x + yd(x)$ for all $x, y \in \mathcal{A}$. Set $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for $x, y \in \mathcal{A}$. A linear map $d : \mathcal{A} \to \mathcal{A}$ is called *Lie derivation* if

E-mail address: ywang2004@126.com.

http://dx.doi.org/10.1016/j.laa.2014.06.029

$d([x,y]) = [d(x),y] + [x,d(y)]$ for all $x,y \in \mathcal{A}$. Similarly, a linear map $d : \mathcal{A} \to \mathcal{A}$ is called *Jordan derivation* if $d(x \circ y) = d(x) \circ y + x \circ d(y)$ for all $x,y \in \mathcal{A}$. Set $p_1(x) = x$ and

$$p_n(x_1, x_2, \dots, x_n) = \big[p_{n-1}(x_1, x_2, \dots, x_{n-1}), x_n\big]$$

for an integer $n \geq 2$. Thus, $p_2(x_1,x_2) = [x_1,x_2]$, $p_3(x_1,x_2,x_3) = [[x_1,x_2],x_3]$, etc. More generally, a linear map $\varphi : \mathcal{A} \to \mathcal{A}$ is called a *Lie n-derivation* if

$$\varphi\big(p_n(x_1, x_2, \dots, x_n)\big) = \sum_{i=1}^{n} p_n\big(x_1, \dots, x_{i-1}, \varphi(x_i), x_{i+1}, \dots, x_n\big)$$

for all $x_1, x_2, \dots, x_n \in \mathcal{A}$. Lie n-derivations were introduced by Abdullaev [1], where the form of Lie n-derivations of a certain von Neumann algebra was described. A Lie 3-derivation is said to be a *Lie triple derivation* (see [5] for details).

Lie (triple) derivations on several classes of algebras have been investigated by many authors (see [2,3,7,9–14,16,17]). In most cases it turns out that a Lie (triple) derivation $\Delta : \mathcal{A} \to \mathcal{A}$ is of the so-called standard form, i.e., the sum $\Delta = d + \gamma$, where $d : \mathcal{A} \to \mathcal{A}$ is a derivation and $\gamma : \mathcal{A} \to Z(\mathcal{A})$ is a central linear map. In 2012, Benkovič and Eremita [5] gave a sufficient condition for a Lie n-derivation on triangular algebras to be of the standard form, which has been generalized to generalized matrix algebras in [18].

We now assume that $\mathcal{A}$ is an algebra with an idempotent $e \neq 0,1$ and let f denote the idempotent $1-e$. In this case $\mathcal{A}$ can be represented in the so-called Peirce decomposition form

$$\mathcal{A} = e\mathcal{A}e + e\mathcal{A}f + f\mathcal{A}e + f\mathcal{A}f,$$

where $e\mathcal{A}e$ and $f\mathcal{A}f$ are subalgebras with unitary e and f, respectively, $e\mathcal{A}f$ is an $(e\mathcal{A}e, f\mathcal{A}f)$-bimodule and $f\mathcal{A}e$ is an $(f\mathcal{A}f, e\mathcal{A}e)$-bimodule. We will assume that $\mathcal{A}$ satisfies

$$\begin{aligned} exe \cdot e\mathcal{A}f = \{0\} = f\mathcal{A}e \cdot exe \quad &\text{implies} \quad exe = 0, \\ e\mathcal{A}f \cdot fxf = \{0\} = fxf \cdot f\mathcal{A}e \quad &\text{implies} \quad fxf = 0 \end{aligned} \tag{1}$$

for all $x \in \mathcal{A}$. The property (1) was introduced by Benkovič and Širovnik [6]. Some specific examples of unital algebras with nontrivial idempotents having the property (1) are triangular algebras, matrix algebras, and prime algebras with nontrivial idempotents (see [4,6] for details). We remark that generalized matrix algebras can be viewed as special unital algebras with nontrivial idempotents having the property (1) (see [19] for details).

We say that a Jordan derivation $\delta : \mathcal{A} \to \mathcal{A}$ is a *singular Jordan derivation* according to the decomposition $\mathcal{A} = e\mathcal{A}e + e\mathcal{A}f + f\mathcal{A}e + f\mathcal{A}f$, if

$$\delta(e\mathcal{A}e + f\mathcal{A}f) = 0, \qquad \delta(e\mathcal{A}f) \subseteq f\mathcal{A}e, \qquad \delta(f\mathcal{A}e) \subseteq e\mathcal{A}f.$$