Linear Algebra and its Applications 458 (2014) 534–541

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the similarity of tensors ☆

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ARTICLE INFO

Article history:
Received 24 October 2013
Accepted 15 June 2014
Available online 3 July 2014
Submitted by R. Brualdi

MSC:
15A69
15A72

Keywords:
Tensor
Product
Similarity
Hypergraph

ABSTRACT

In this paper, we characterize all similarity relations when $m \geq 3$, obtain some interesting properties which are different from the matrix case ($m = 2$), and show that some of the well-known results of matrices cannot be extended to tensors of order $m \geq 3$.

1. Introduction

Since the work of Qi [4] and Lim [2], the study of tensors, the spectra of tensors (and hypergraphs) and their various applications has attracted much attention and interest.

☆ P. Yuan's research is supported by the NSF of China (Grant No. 11271142) and the Guangdong Provincial Natural Science Foundation (Grant No. S2012010009942), L. You's research is supported by the Zhujiang Technology New Star Foundation of Guangzhou (Grant No. 2011J2200090) and Program on International Cooperation and Innovation, Department of Education, Guangdong Province (Grant No. 2012gjhz0007).

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http://dx.doi.org/10.1016/j.laa.2014.06.026

An order m dimension n tensor $\mathbb{A} = (a_{i_1 i_2 \ldots i_m})_{1 \leq i_j \leq n \, (j=1,\ldots,m)}$ over the complex field $\mathbb{C}$ is a multidimensional array with all entries $a_{i_1 i_2 \ldots i_m} \in \mathbb{C}$ $(i_1, \ldots, i_m \in [n] = \{1, \ldots, n\})$. The majorization matrix $M(\mathbb{A})$ of the tensor $\mathbb{A}$ is defined as $(M(\mathbb{A}))_{ij} = a_{ij\ldots j}$, $(i, j \in [n])$ by Pearson [3]. The unit tensor of order m and dimension n is the tensor $\mathbb{I} = (\delta_{i_1,i_2,\ldots,i_m})$ with

$$\delta_{i_1,i_2,\ldots,i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \ldots = i_m; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbb{A}$ (and $\mathbb{B}$) be an order $m \geq 2$ (and $k \geq 1$), dimension n tensor, respectively. Recently, Shao [5] defined a general product $\mathbb{AB}$ to be the following tensor $\mathbb{D}$ of order $(m-1)(k-1)+1$ and dimension n:

$$d_{i\alpha_1 \ldots \alpha_{m-1}} = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} b_{i_2\alpha_1} \ldots b_{i_m\alpha_{m-1}} \quad \left(i \in [n],\ \alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}\right).$$

The tensor product possesses a very useful property: the associative law ([5], Theorem 1.1). With the general product, the following definition of the similarity relation of two tensors was proposed by Shao [5].

Definition 1.1. (See [5], Definition 2.3.) Let $\mathbb{A}$ and $\mathbb{B}$ be two order m dimension n tensors. Suppose that there exist two matrices P and Q of order n with $P\mathbb{I}Q = \mathbb{I}$ such that $\mathbb{B} = P\mathbb{A}Q$, then we say that the two tensors are similar.

It is easy to see that the similarity relation is an equivalent relation, and similar tensors have the same characteristic polynomials, and thus they have the same spectrum (as a multiset). For example, the permutation similarity and the diagonal similarity (see also [5–7]) are two special kinds of the similarity of tensors.

Definition 1.2. (See [5].) Let $\mathbb{A}$ and $\mathbb{B}$ be two order m dimension n tensors. We say that $\mathbb{A}$ and $\mathbb{B}$ are permutational similar, if there exists some permutation matrix P of order n such that $\mathbb{B} = P\mathbb{A}P^T$, where $\sigma \in S_n$ is a permutation on the set $[n]$ and $P = P_\sigma = (p_{ij})$ is the corresponding permutation matrix of σ with $p_{ij} = 1 \Leftrightarrow j = \sigma(i)$.

Definition 1.3. (See [5], Definition 2.4.) Let $\mathbb{A}$ and $\mathbb{B}$ be two order m dimension n tensors. We say that $\mathbb{A}$ and $\mathbb{B}$ are diagonal similar, if there exists some invertible diagonal matrix D of order n such that $\mathbb{B} = D^{-(m-1)}\mathbb{A}Q$.

About the matrices P and Q in Definition 1.1, [5] showed the following proposition.

Proposition 1.4. (See [5], Remark 2.1.) *If P and Q are two matrices of order n with $P\mathbb{I}Q = \mathbb{I}$, where $\mathbb{I}$ is the order m dimension n unit tensor, then both P and Q are invertible matrices.*