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On graphs with at least three distance eigenvalues less than -1 ☆

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ABSTRACT

Let G be a connected graph with order n and $D(G)$ be the distance matrix of G. Suppose that $\lambda_1(D) \geq \lambda_2(D) \geq \cdots \geq \lambda_n(D)$ are the D-eigenvalue of G. In this paper, we show that $\lambda_{n-1}(D(G)) \leq -1$ if $n \geq 4$ and $\lambda_{n-2}(D(G)) \leq -1$ if $n \geq 7$. We also characterize all connected graphs with $\lambda_{n-1}(D(G)) = -1$, moreover it is shown that these graphs are determined by their distance spectra.

1. Introduction

Unless stated otherwise, we follow [1] for terminology and notations, and we consider finite connected simple graphs. In particular, denote $V(G) = \{v_1, \cdots, v_n\}$ the vertex set

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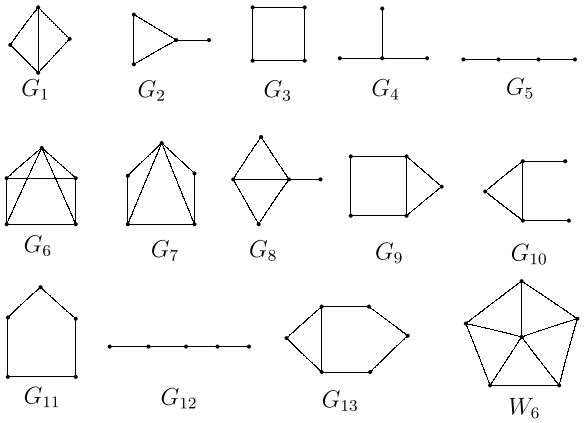

Fig. 1. Graphs $G_1 - G_{13}$ and W_6.

of G, $E(G)$ the edge set of G. For a graph $G = (V, E)$, two vertices are called *adjacent* if they are connected by an edge. $|V(G)|$ is always called the order of G and $|E(G)|$ is always called the size of G. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph induced by S. A graph is $\{H_1, H_2, \cdots, H_k\}$-free if it contains no induced subgraph isomorphic to any H_i for $i = 1, \cdots, k$.

Let G be a connected graph. The distance matrix $D(G)$ is nonnegative and irreducible, so the eigenvalues of $D(G)$ are real and we can order the eigenvalues as $\lambda_1(D) \geq \lambda_2(D) \geq \cdots \geq \lambda_n(D)$. The largest eigenvalue of $D(G)$ is called the *distance spectral radius* of G, denoted by $\lambda_1(D)$ and $\lambda_{n-1}(D)$ is called the second smallest D-eigenvalue. The positive unit eigenvector corresponding to $\lambda_1(D)$ is called the *Perron vector* of $D(G)$. There are a lot of papers about the distance spectral radius, see [4,5,8,9,11,12].

The complete product $G_1 \vee G_2$ of graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2. Let K_n denote the complete graph on n vertices, and let $W_n = K_1 \vee C_{n-1}$ denote the wheel graph. Let $K_{n_1,\cdots,n_k}$ denote the complete k-partite graph. Lin et al. [7] showed that $\lambda_n(D(G)) = -2$ if and only if $G \cong K_{n_1,\cdots,n_k}$, and the result is independently proved by Yu [10] in a different way. Let $K^r_{s,t} = K_r \vee (K_s \cup K_t)$ with $r \geq 1$. We first give the following two theorems.

Theorem 1.1. *Let G be a connected graph with order n and D be the distance matrix of G. If $n \geq 4$, then $\lambda_{n-1}(D) \leq -1$ and the equality holds if and only if $G \cong K^r_{s,t}$ with $r \geq 1$.*

Theorem 1.2. *Let G be a connected graph with order n and D be the distance matrix of G. (See Fig. 1.)*

(1) If $n = 4$, then $\lambda_2(D) \leq -1$ unless G is one of $\{G_i \mid i = 1, \cdots, 5\}$.
(2) If $n = 5$, then $\lambda_3(D) \leq -1$ unless G is one of $\{G_i \mid i = 6, \cdots, 12\}$.
(3) If $n = 6$, then $\lambda_4(D) \leq -1$ unless either $G \cong W_6$ or $G \cong G_{13}$.
(4) If $n \geq 7$, then $\lambda_{n-2}(D) \leq -1$.