Linear Algebra and its Applications 458 (2014) 573–588

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

More accurate weak majorization relations for the Jensen and some related inequalities

Mario Krnić [a,*], Josip Pečarić [b]

[a] University of Zagreb, Faculty of Electrical Engineering and Computing, Unska 3, 10000 Zagreb, Croatia
[b] University of Zagreb, Faculty of Textile Technology, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia

A R T I C L E   I N F O

Article history:
Received 1 April 2014
Accepted 23 June 2014
Available online 9 July 2014
Submitted by R. Brualdi

MSC:
47A30
47B15
15A45
26A51

Keywords:
Convex function
Jensen inequality
Eigenvalue
Weak majorization
Unitarily invariant norm

A B S T R A C T

Motivated by results of Aujla and Silva [3], we give several more precise weak majorization and eigenvalue inequalities for some matrix versions of the famous Jensen inequality with regard to a convexity. Our main results are then applied to log convex functions. As an application, we obtain refinements of some well-known matrix inequalities.

1. Introduction

Throughout this article $\mathcal{M}_n$ is the algebra of all $n \times n$ complex matrices and $\mathcal{H}_n$ stands for the set of all Hermitian matrices in $\mathcal{M}_n$. For an interval $J \subseteq \mathbb{R}$, we denote by

* Corresponding author.
E-mail addresses: mario.krnic@fer.hr (M. Krnić), pecaric@element.hr (J. Pečarić).

http://dx.doi.org/10.1016/j.laa.2014.06.037

$\mathcal{H}_n(J)$ the set of all Hermitian matrices in $\mathcal{M}_n$ whose spectrum is contained in J. We denote by $\mathcal{S}_n$, the set of all positive semi-definite matrices in $\mathcal{M}_n$, while $\mathcal{P}_n$ stands for the set of all positive definite matrices in $\mathcal{M}_n$. For column vectors $x, y \in \mathbb{C}^n$ their inner product is denoted by $\langle x, y\rangle = y^* x$.

For Hermitian matrices A and B we define an operator order, i.e. $A \le B$ if $B - A \in \mathcal{S}_n$. Further, for $A \in \mathcal{H}_n$ we denote by $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$ the eigenvalues of A arranged in a decreasing order with their multiplicities counted. The notation $\lambda(A)$ stands for the row vector $(\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A))$. The eigenvalue inequality $\lambda(A) \le \lambda(B)$ means that $\lambda_j(A) \le \lambda_j(B)$ for all $1 \le j \le n$. The weak majorization inequality $\lambda(A) \prec_w \lambda(B)$ means $\sum_{j=1}^k \lambda_j(A) \le \sum_{j=1}^k \lambda_j(B)$, $k = 1, 2, \dots, n$. The above three kinds of ordering satisfy $A \le B \Rightarrow \lambda(A) \le \lambda(B) \Rightarrow \lambda(A) \prec_w \lambda(B)$. Note that the first implication is the Weyl monotonicity theorem (see, e.g. [5, p. 63]), while the second holds trivially.

A real valued function defined on the interval J is called convex if

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y), \tag{1}$$

for all $0 \le t \le 1$ and $x, y \in J$. If the sign of inequality (1) is reversed, f is called a concave function. One of the most important inequalities in connection with a convex function $f : J \to \mathbb{R}$ is the famous Jensen inequality which asserts that

$$f\left(\sum_{i=1}^m p_i x_i\right) \le \sum_{i=1}^m p_i f(x_i), \tag{2}$$

where $\sum_{i=1}^m p_i = 1$, $p_i \ge 0$, and $x_i \in J$, $i = 1, 2, \dots, m$. For a comprehensive inspection on convex functions, its properties and the corresponding inequalities, the reader is referred to [14].

On the other hand, $f : J \to \mathbb{R}$ is operator convex if

$$f\big(tA + (1-t)B\big) \le tf(A) + (1-t)f(B), \tag{3}$$

for all $0 \le t \le 1$ and $A, B \in \mathcal{H}_n(J)$. Recall that for Hermitian matrix $H \in \mathcal{H}_n(J)$, $f(H)$ is defined by familiar functional calculus.

One of the numerous operator versions of the Jensen inequality asserts that if $f : J \to \mathbb{R}$ is operator convex function such that $0 \in J$ and $f(0) \le 0$, then

$$f\big(X^* A X\big) \le X^* f(A) X \tag{4}$$

holds for all $A \in \mathcal{H}_n(J)$ and contractions $X \in \mathcal{M}_n$. Recall that $X \in \mathcal{M}_n$ is called contraction if $\|X\| \le 1$, where $\|X\|$ is the spectral norm. For some related versions of the Jensen operator inequality, the reader is referred to [7].

Some ten years ago, Aujla and Silva [3], proved that if $f : J \to \mathbb{R}$ is a convex function, then the eigenvalues of $f(tA + (1-t)B)$ are weakly majorized by the eigenvalues of