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Three-by-three correlation matrices: its exact shape and a family of distributions

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ABSTRACT

We give a novel and simple convex construction of three-by-three correlation matrices. This construction reveals the exact shape of the volume for these matrices: it is a tetrahedron point-wise transformed through the sine function. Hence the space of three-by-three correlation matrices is isomorphic to the standard three-simplex, and the matrices can be sampled by placing distributions on the three-simplex. This gives densities on the matrices that are flexible and easily interpreted; these will be useful in Bayesian analysis of correlation matrices. Examples using Dirichlet distributions are provided. We show the uniqueness of the construction, and we also prove that there is no parallel construction for higher order correlation matrices.

1. Introduction

The correlation is one of the most easily understood and widely used statistics. In the case of two random variables X_1 and X_2, it is clear that the correlation coefficient can take any value in $[-1, 1]$. Therefore, it is straightforward to sample the correlation

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matrix for X_1 and X_2. Unfortunately, such simplicity is lost when there is a third random variable X_3 so that the joint correlation matrix is now

$$C \stackrel{\text{def}}{=} \begin{pmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_3 \\ r_2 & r_3 & 1 \end{pmatrix}.$$

In this case, the three correlations, r_1, r_2 and r_3, are dependent on each other, and it is insufficient to only restrict the range of each to $[-1, 1]$. There have been numerous studies on the structure of C [6,1,8], and one commonly used criterion is the non-negativity of the determinant of C:

$$\det(C) = 1 + 2r_1r_2r_3 - r_1^2 - r_2^2 - r_3^2 \geq 0. \tag{1}$$

However, this constraint is not constructive. In this paper, we shall give a simple convex construction for the three correlations. The construction is of practical interest because it immediately provides a family of distributions on three-by-three correlation matrices that is induced by distributions on the standard three-simplex $\{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4 \mid \sum_{i=1}^4 \alpha_i = 1 \text{ and } \alpha_i \geq 0 \text{ for all } i\}$. The construction also has pedagogical value in furthering our understanding of such correlation matrices. In particular, it provides a precise definition to the *elliptical tetrahedron* investigated by Rousseeuw and Molenberghs [9]. We will also show that it is not possible to construct higher-order correlation matrices in the same manner.

The next section will give the construction of three-by-three correlation matrices. The construction provides two new defining characteristics of the correlation matrices, which will be stated in Section 3 in addition to existing ones on the determinant and the partial correlations. In Section 4, we will introduce a family of distributions on three-by-three correlation matrices using the construction. These distributions will be useful in Bayesian analysis of such matrices. We will show that the construction is not extensible to higher order matrices in Section 5. Section 6 will conclude the paper. Throughout, $\vec{1}$ is the vector of ones of the appropriate dimension within the context.

2. Construction

2.1. Orthant probabilities

It is known that the upper orthant probability $Pr(X_1 > 0, X_2 > 0, X_3 > 0)$ of a trivariate centred normal distribution with correlations r_1, r_2 and r_3 is

$$P \stackrel{\text{def}}{=} \frac{1}{8} + \frac{1}{4\pi}(\theta_1 + \theta_2 + \theta_3),$$

where $\theta_i \stackrel{\text{def}}{=} \arcsin r_i$, $i = 1, 2, 3$ [4, Eq. 42]. The maximum and minimum values of the orthant probability are half and zero respectively, both attained with degenerate normal