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The majorization theorem of extremal pseudographs ☆

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A B S T R A C T

A pseudograph is a graph in which both loops and multiple edges are permitted. Suppose $\pi = (d_1, d_2, \dots, d_n)$ and $\pi' = (d'_1, d'_2, \dots, d'_n)$ are two positive non-increasing degree sequences, we write $\pi \lhd \pi'$ if and only if $\pi \neq \pi'$, $\sum_{i=1}^n d_i = \sum_{i=1}^n d'_i$, and $\sum_{i=1}^j d_i \leq \sum_{i=1}^j d'_i$ for all $j = 1, 2, \dots, n$. Let $\Gamma(\pi)$ be the class of connected undirected pseudographs with degree sequence π. Let $\rho(G)$ and $\mu(G)$ be the spectral radius and signless Laplacian spectral radius of G, respectively. In this paper, the extremal pseudographs with the largest (respectively, signless Laplacian) spectral radii in $\Gamma(\pi)$ are characterized. Furthermore, we show that if $\pi \lhd \pi'$, G and G' are the pseudographs with the largest (respectively, signless Laplacian) spectral radii in $\Gamma(\pi)$ and $\Gamma(\pi')$, respectively, then $\rho(G) < \rho(G')$ and $\mu(G) < \mu(G')$.

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1. Introduction

A *pseudograph* is a graph in which both loops and multiple edges are permitted. Thus, a simple graph is also a pseudograph, but not vice versa. Throughout this paper, $G = (V(G), E(G))$ is a connected undirected pseudograph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G)$. The degree of a vertex v of G, denoted $d(v)$, is the number of edges of G incident with v, each loop counting as two edges. Denote $N_G(v)$ the neighbor set of vertex v in G. Let $\Upsilon_{uv}(G)$ be the number of edges joining vertices u and v in G. In particular, $\Upsilon_{uu}(G)$ indicates the number of loops incident with vertex u in G. If there is no confusion, we simplify $N_G(v)$ and $\Upsilon_{uv}(G)$ as $N(v)$ and Υ_{uv}, respectively.

The adjacency matrix of G is an $n \times n$ matrix $A(G) = (a_{ij})$, where a_{ij} is the number of edges joining vertices v_i and v_j, each loop counting as two edges.

In the following, let G be a graph with n vertices and let $f \neq 0$ be a column vector defined on $V(G)$, i.e., $f = (f(v_1), f(v_2), ..., f(v_n))^T$. If there exists a real number q such that for each $u \in V(G)$,

$$qf(u) = \big(A(G)f\big)(u) = \sum_{v \in N(u) \setminus \{u\}} \Upsilon_{uv} f(v) + 2\Upsilon_{uu} f(u), \tag{1.1}$$

then q is called an eigenvalue of $A(G)$. The *spectral radius* of G, denoted $\rho(G)$, is the largest eigenvalue of $A(G)$.

The *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G. If there exists a real number p such that for each $u \in V(G)$,

$$pf(u) = \big(Q(G)f\big)(u) = \sum_{v \in N(u) \setminus \{u\}} \Upsilon_{uv}\big(f(u) + f(v)\big) + 4\Upsilon_{uu} f(u), \tag{1.2}$$

then p is called an eigenvalue of $Q(G)$. It is easy to see that $Q(G)$ is positive semidefinite [4] and hence we use $\mu(G)$ to denote the largest eigenvalue of $Q(G)$. Hereafter, $\mu(G)$ is called the *signless Laplacian spectral radius* of G.

When G is a connected simple graph, the *Laplacian matrix* of G is defined as $L(G) = D(G) - A(G)$. When G is a connected simple graph, it is well known that [13] $L(G)$ is also positive semidefinite so that we can use $\lambda(G)$ to denote the largest eigenvalue of $L(G)$. Hereafter, $\lambda(G)$ is called the *Laplacian spectral radius* of G.

By Eqs. (1.1) and (1.2), we have $\mathcal{R}_{A(G)}(f) = f^T A(G) f = 2\sum_{uv \in E(G)} \Upsilon_{uv} f(u) f(v)$ and $\mathcal{R}_{Q(G)}(f) = f^T Q(G) f = \sum_{uv \in E(G)} \Upsilon_{uv}(f(u) + f(v))^2$. By the famous Rayleigh–Ritz Theorem (see e.g. [5, pp. 172–178]) it follows that

Proposition 1.1. *(See [5].) Let G be a connected undirected pseudograph, and let f be an n-tuple unit vector on $V(G)$. Then,*

$$\rho(G) \geq \mathcal{R}_{A(G)}(f) \quad \textit{and} \quad \mu(G) \geq \mathcal{R}_{Q(G)}(f),$$