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On square roots and norms of matrices with symmetry properties

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The present work concerns the algebra of semi-magic square matrices. These can be decomposed into matrices of specific rotational symmetry types, where the square of a matrix of pure type always has a particular type. We examine the converse problem of categorising the square roots of such matrices, observing that roots of either type occur, but only one type is generated by the functional calculus for matrices. Some explicit construction methods are given. Moreover, we take an observation by N.J. Higham as a motivation for determining bounds on the operator p-norms of semi-magic square matrices.

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1. Introduction

Traditionally a square matrix $M = (m_{ij})_{i,j\in\{1,\dots,n\}}$, which satisfies the constant row and column sum symmetry condition

$$\sum_{j=1}^{n} m_{ij} = c \quad (1 \le i \le n), \qquad \sum_{i=1}^{n} m_{ij} = c \quad (1 \le j \le n), \tag{s1}$$

for some constant number c, is called a *semi-magic square* with weight c/n (*SMS* or *type S matrix* for short). If all the entries of M are non-negative, then we can write $M = (1/c)H$ with H a doubly-stochastic matrix.

If, in addition to condition (s1), M also satisfies the associated pairwise symmetry condition

$$m_{ij} + m_{(n+1-i)(n+1-j)} = 2c/n \quad (1 \le i,j \le n), \tag{s2}$$

then M is called an *associated magic square* [1] with weight c/n, (*AMS* or *type A matrix* for short). In contrast, if M satisfies condition (s1) and the balanced pairwise symmetry condition

$$m_{ij} = m_{(n+1-i)(n+1-j)} \quad (1 \le i,j \le n), \tag{s3}$$

then we say that M is a *balanced magic square* with weight c/n (*BMS* or *type B matrix* for short). Hence all type A and type B matrices are of type S; a matrix which is simultaneously type A and type B must have every entry the same (see Lemma 2.3).

In [13] the multiplicative properties of these matrix types in the 3×3 case are considered. For general n, the result is as follows (see [7] and [8], Lemma 3.1).

Lemma 1.1. *Let M and N be $n \times n$ type* S *matrices with respective weights z and w. Then it follows that*

(1) *MN is type* S *with weight nzw.*
(2) *If M and N are both type* A *or both type* B*, then MN is type* B.
(3) *If M is type* A *and N is type* B*, then MN and NM are type* A.
(4) *If M is invertible, then M^{-1} is type* S *with weight $1/n^2z$.*
(5) *If M is type* B *and invertible, then M^{-1} is type* B.
(6) *If M is type* A *and invertible, then M^{-1} is type* A.

Remark. It follows from the above lemma that if M is type A then M^r is type A for all positive odd r and type B for all positive even r. Similarly, if M is type B then M^r is type B for all positive r. (Clearly, if M is also non-singular, then this holds for all $r \in \mathbb{Z}$.) It is also clear from Lemma 1.1 (2) that a type A matrix cannot have any square root of type A or B.